Enumerative Properties of Rooted Circuit Maps

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Abstract

In 1966 Barnette introduced a set of graphs, called *circuit graphs*, which are obtained from 3-connected planar graphs by deleting a vertex. Circuit graphs and 3-connected planar graphs share many interesting properties which are not satisfied by general 2-connected planar graphs. Circuit graphs have nice closure properties which make them easier to deal with than 3-connected planar graphs for studying some graph-theoretic properties. In this paper, we study some enumerative properties of circuit graphs. For enumeration purpose, we define rooted circuit maps and compare the number of rooted circuit maps with those of rooted 2-connected planar maps and rooted 3-connected planar maps.

1 Introduction

In 1966 Barnette introduced a set of graphs, called *circuit graphs*, which are obtained from 3-connected planar graphs by deleting a vertex. Circuit graphs and 3-connected planar graphs share many interesting properties which are not satisfied by general 2-connected planar graphs. For example, circuit graphs have spanning trees with maximum degree at most 3 (called a *3-tree*), and they contain closed walks visiting each vertex once or twice [2, 12, 13]. Very recently Nakamoto et al [17] showed that every circuit graph contains a 3-tree with few vertices of degree 3. It is also known that circuit graphs contain long cycles [15, 9, 14] and they contain 2-connected spanning trees with small maximum degree [3, 11].

For enumeration purpose, it will be convenient to allow multiple edges in our graphs. Graphs without multiple edges will be called *simple graphs*. Recall that a 2-separation of a graph G is a pair (H, K) of subgraphs of G such that $|E(H)| \geq 2$, $|E(K)| \geq 2$, $H \cup K = G$, and $H \cap K$ is the graph with two isolated vertices. The following definition of circuit graphs, in the case of no multiple edges, was given in [12].

Definition 1. A *circuit graph* is an ordered pair (G, C) such that

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(1) G is a 2-connected graph and C is a cycle in G;

(2) there is an embedding of G in the plane such that C bounds a face;

(3) if (H, K) is a 2-separation of G, then $C \not\subseteq H$ and $C \not\subseteq K$.

It was shown in [12] that the above definition of circuit graphs, in the case of no multiple edges, is equivalent to Barnette's definition of circuit graphs. It is easy to see, using the above definition, that if P is a set of parallel edges with common end vertices in a circuit graph (G, C), then $|P \setminus E(C)| \leq 1$. On one hand, as we shall see below, allowing multiple edges makes the enumeration task easier; on the other hand, multiple edges are usually allowed in enumerative map theory.

For enumeration purpose, we shall adopt the following definition.

Definition 2. A (*simple*) *circuit map* is a (simple) circuit graph (G, C) embedded in the plane so that C bounds the exterior face. A circuit map is rooted by specifying a vertex (called the *root vertex*) in C and an edge (called the *root edge*) in C incident with the root vertex. The exterior face is called the *root face*.

We note that if e_1, e_2 are parallel edges in a circuit map C, then e_1 and e_2 must bound a digon (a face of degree 2) and at least one of them must be in C.

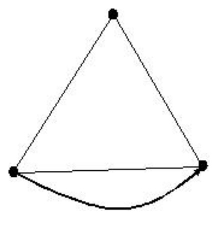

Figure 1: A circuit map with multiple edges

Much work has been done during 1960's by Tutte and his students on enumerating rooted planar maps (See, e.g., [7] for a survey). It is known that the number $M_{k,n}$ of rooted k-connected planar maps with n edges satisfies the following pattern

$$M_{k,n} \sim K_k n^{-5/2} R_k^n,$$

where

$$\begin{array}{lll} K_1 = \frac{2}{\sqrt{\pi}}, & K_2 = \frac{2}{9\sqrt{3\pi}}, & K_3 = \frac{2}{243\sqrt{\pi}}, \\ R_1 = 12, & R_2 = 27/4, & R_3 = 4. \end{array}$$

The exponent of n in the above asymptotic expression is usually called the *critical exponent*, and it is $-5/2$ for a typical family of rooted planar maps.

Let $C_{n,k}$ ($\bar{C}_{n,k}$) be the number of rooted (simple) circuit maps with n edges and root face degree k. Then $C_n = \sum_k C_{n,k}$ ($\bar{C}_n = \sum_k \bar{C}_{n,k}$) is the number of rooted (simple) circuit maps with n edges. The main result of the paper is the following.

Theorem 1 *(i) Let* $u_0 = \dfrac{11}{21} + \dfrac{10}{21}(692 + 84\sqrt{69})^{-1/3} - \dfrac{1}{42}(692 + 84\sqrt{69})^{1/3}$,

$$\begin{aligned} R &= \frac{(1-u_0)^2}{u_0(1-2u_0)} \doteq 4.08, \\ a_1 &= \frac{u_0(1-3u_0+u_0^2)}{(1-3u_0)(1-u_0)^6}\sqrt{u_0(1-u_0)(1-2u_0)(1-3u_0)(1-2u_0-u_0^2+u_0^3)(6-22u_0+21u_0^2)} \\ &\doteq 0.147041065. \end{aligned}$$

Then $C_n \sim \frac{a_1}{4\sqrt{\pi}} n^{-3/2} R^n$.

(ii) $\bar{C}_n \sim \frac{6}{25\sqrt{\pi}} n^{-5/2} 4^n$.

(iii) Let $q_k = \lim_{n\to\infty} \frac{\bar{C}_{n,k}}{\bar{C}_n}$ *be the limiting probability that a random rooted circuit map has root face degree* k*. Then*

$$\sum_{k\geq 3} q_k Z^k = \frac{Z(27Z^3 - 315Z^2 + 800Z - 500 + (5-Z)(10-9Z)\sqrt{(10-Z)(10-9Z)})}{8(10-9Z)\sqrt{(10-Z)(10-9Z)}},$$

and

$$q_k \sim \frac{25}{162}\sqrt{\frac{k}{2\pi}}(9/10)^k, \ \text{as } k \to \infty.$$

(iv) For any positive constant ϵ*,*

$$\frac{\bar{C}_{n,k}}{\bar{C}_n} = O\left(\left(\frac{9+\epsilon}{10}\right)^k\right),$$

uniformly for all k *as* $n \to \infty$*.*

It is interesting to note that the critical exponent for rooted circuit maps is $-3/2$, not the typical $-5/2$.

2 The bijection between rooted maps and rooted quadrangulations

The following bijection Φ between rooted maps and rooted quadrangulations is well known (See, e.g., [16]). Insert a vertex inside each face of a map M and join it to each vertex on the face to subdivide the face into triangles. Removing the edges of M gives the quadrangulation $\Phi(M)$ whose vertices are partitioned into two independent sets: the vertices of M and the new vertices. The rooting of $\Phi(M)$ can be chosen in the following canonical way: the vertex f_0 inside the root face of M is the root vertex of $\Phi(M)$, the edge joining f_0 and the root vertex of M is the root edge of $\Phi(M)$.

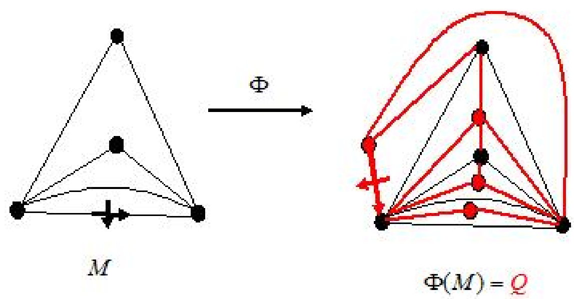

Figure 2: Bijection between rooted maps and rooted quadrangulations

A quadrangle is called *separating* in $\Phi(M)$ if there are vertices both in its interior and in its exterior. A quadrangle is called *root-separating* if it contains vertices in its interior and the root vertex is in its exterior.

It is easy to check that the bijection Φ satisfies the following

Proposition 1:

- M is 2-connected if and only if $\Phi(M)$ has no multiple edges,
- M is 3-connected if and only if $\Phi(M)$ has no multiple edges and no separating quadrangles.
- M is a circuit map if and only if $\Phi(M)$ has no multiple edges and no root-separating quadrangles.

Let $Q_{i,j,k}$ be the number of rooted quadrangulations with no multiple edges, and with i red vertices, j blue vertices and root vertex degree k. Define the generating function

$$Q(x,y,z) = \sum_{i,j,k\geq 2} Q_{i,j,k} x^{i-1} y^{j-1} z^k.$$

By Proposition 1 and [8, (3.8)], we have

$$\begin{aligned} & Q^2(x,y,z) + ((1-z)(1-xz) + yz - zQ(x,y,1))\, Q(x,y,z) \\ = \;& yz^2(x(1-z) + Q(x,y,1)). \end{aligned} \tag{1}$$

Also from [8], we have

$$Q(x,y,1) = uv(1-u-v), \tag{2}$$

where u and v are unique power series in x and y defined by

$$x = u(1-v)^2, \quad y = v(1-u)^2. \tag{3}$$

We note that $Q(x,y,z)$ is now determined by (1).

Let $C_{i,j,k}$ be the number of rooted circuit maps with i vertices, j faces, and root face degree k, and let

$$C(X,Y,Z) = \sum C_{i,j,k} X^{i-1} Y^{j-1} Z^k.$$

Define $\bar{C}(X,Y,Z)$ analogously for rooted simple circuit maps with at least 3 vertices. We have

Theorem 2 *Let $X = Q(x,y,1)/y,\ Y = Q(x,y,1)/x,\ Z = xyz/Q(x,y,1)$. Then*

$$\begin{aligned} C(X,Y,Z) &= Q(x,y,z), & (4)\\ \bar{C}(X,Y,Z(1+Y)) &= C(X,Y,Z) - XY(1+Y)Z^2. & (5) \end{aligned}$$

Proof: For any rooted quadrangulation Q, call a root-separating quadrangle *maximal* if it is not inside another root-separating quadrangle. It is easy to see that the interiors of maximal root-separating quadrangles are pairwise disjoint. Therefore, removing all vertices and edges in the interior of each maximal root-separating quadrangle yields a root-simple quadrangulation, and this process can be reversed by replacing each face of a root-simple quadrangulation, that is not incident with the root vertex, with an arbitrary rooted quadrangulation. Similar to $Q(x,y,z)$ for all quadrangulations without multiple edges, we let $\bar{Q}(x,y,z)$ be the generating function for root-simple quadrangulations. In a quadrangulation with $i+j$ vertices and root vertex degree k, we note that there are $i+j-2$ faces (by Euler's formula) in total, and hence there are $i+j-2-k$ faces not incident with the root vertex. It follows from the above correspondence between the two families of rooted quadrangulations that

$$\begin{aligned} Q(x,y,z) &= \sum \bar{Q}_{i,j,k} x^{i-1} y^{j-1} z^k \left(Q(x,y,1)/xy\right)^{i+j-2-k} \\ &= \bar{Q}(Q(x,y,1)/y, Q(x,y,1)/x, xyz/Q(x,y,1)). \end{aligned}$$

Now (4) follows from Proposition 1. We note that there are exactly two rooted circuit maps with two vertices: the one with two parallel edges and the other one with three parallel edges. Also rooted circuit maps with more than two vertices can be generated from rooted simple circuit maps with more than two vertices by replacing some edges on the root face with a digon. This gives (5). ∎

3 Proof of Theorem 1

It follows from Euler's formula for planar maps that

$$C(X,X,Z) = \sum_{n,k} C_{n,k} X^n Z^k, \tag{6}$$

$$\bar{C}(X,X,Z) = \sum_{n,k} \bar{C}_{n,k} X^n Z^k. \tag{7}$$

Setting $u=v$ in (2) and (3), we obtain

$$x = y = u(1-u)^2, \ Q(x,x,1) = u^2(1-2u). \tag{8}$$

It follows from Theorem 2 that

$$X = Y = \frac{u(1-2u)}{(1-u)^2}. \tag{9}$$

Setting $Z=1$, we obtain from Theorem 2 that

$$z = \frac{(1-2u)}{(1-u)^4}.$$

Thus it follows from (1) and (4) that

$$\begin{aligned} C(X,X,1) &= \frac{u(3u^2-3u+1)(u^3-u^2-2u+1)}{2(1-u)^6} \\ &\quad - \frac{u(u^2-3u+1)}{2(1-u)^6}\sqrt{(u^3-u^2-2u+1)(1-6u+11u^2-7u^3)}. \end{aligned} \tag{10}$$

From (9), we have

$$u = \frac{1-\sqrt{1-4X}}{3-\sqrt{1-4X}}. \tag{11}$$

Substituting it into (10) and expanding $C(X,X,1)$ into power series of X, we obtain

$$C(X,X,1) = X^2 + 2X^3 + 4X^4 + 10X^5 + 27X^6 + \cdots.$$

The correctness of the first few terms are verified by the rooted circuit maps listed in Figure 3.

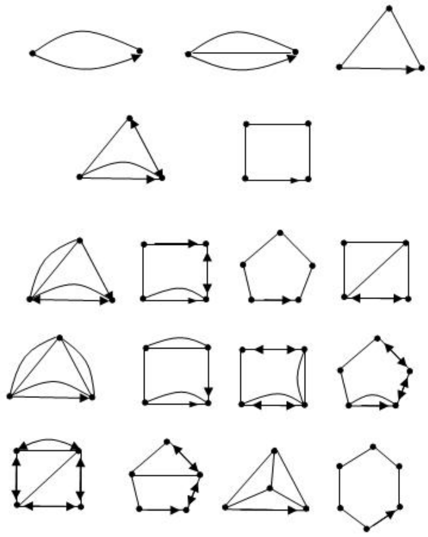

Figure 3: A list of rooted circuit maps with up to 6 edges

Let $X_0 > 0$ be the singularity of $C(X, X, 1)$ closest to the origin. It is easy to see from (11) that $u(X)$ is an increasing function of X for $0 \le X \le 1/4$, and $u(1/4) = 1/3$. It follows from (10) that X_0 corresponds to the smallest value among $u = 1/3$ and the positive roots of

$$(u^3 - u^2 - 2u + 1)(1 - 6u + 11u^2 - 7u^3) = 0.$$

Using Maple, it is easy to check that the smallest positive root of the above equation is $u_0 \doteq 0.301 < 1/3$, which comes from the second factor and is given in Theorem 1(i). Hence

$$R = \frac{1}{X_0} = \frac{(1-u_0)^2}{u_0(1-2u_0)},$$

and the expressions for u_0 and R in Theorem 1 follow immediately.

To obtain the asymptotic expansion of $C(X, X, 1)$ at the singularity X_0, we first note

$$1 - 6u + 11u^2 - 7u^3 = \left(6 - 22u_0 + 21u_0^2\right)(u_0 - u) + O\left((u_0 - u)^2\right), \text{ as } u \to u_0.$$

Then we use (11) to obtain

$$u_0 - u = \frac{(1-u_0)^3}{(1-3u_0)}(X_0 - X) + O\left((X_0 - X)^2\right), \text{ as } X \to X_0.$$

It follows from (10) that, as $X \to X_0$,

$$C(X, X, 1) = c + \frac{-a_1}{2}(1 - X/X_0)^{1/2} + a_2(1 - X/X_0) + O\left((1 - X/X_0)^{3/2}\right)$$

for some constant c, where a_1 is given in part (i) of Theorem 1.

Since $C(X, X, 1)$ is algebraic, it follows from Darboux's lemma (see, e.g., [4]) that

$$C_n \sim \frac{a_1}{2\sqrt{\pi}} n^{-3/2} X_0^{-n},$$

which completes the proof of part (i) of Theorem 1.

To obtain $\bar{C}(X,X,1)$, we set $Z = 1/(1+Y)$. Then it follows from Theorem 2 that

$$z = \frac{1-2u}{(1-u)^2(1-u-u^2)},$$

and

$$\bar{C}(X,X,1) = Q(x,x,z) - \frac{X^2}{1+X}.$$

Using (1) and (9), we obtain

$$Q(x,x,z) = \frac{u^2(1-2u-u^2)}{(1-u-u^2)^2},$$

and hence

$$\bar{C}(X,X,1) = \frac{u^2(1-2u-u^2)}{(1-u-u^2)^2} - \frac{X^2}{1+X},$$

where u is the function of X given in (11).

Using Maple, we obtain the following power series expansion

$$\bar{C}(X,X,1) = X^3 + X^4 + 3X^5 + 7X^6 + \cdots$$

Again the correctness of the first few terms in the above expansion is verified by Figure 3.

Since $1-u-u^2$ is not zero when $0 \le X \le 1/4$, it follows that the dominant singularity of $\bar{C}(X,X,1)$ is $X = 1/4$. We also have the following asymptotic expansion at $X = 1/4$:

$$\bar{C}(X,X,1) + \frac{X^2}{1+X} = 2/25 - (36/125)(1-4X) + (8/25)(1-4X)^{3/2} + O\left((1-4X)^2\right).$$

Again, a simple application of Darboux's lemma gives

$$\bar{C}_n = [X^n]\bar{C}(X,X,1) \sim \frac{8}{25\Gamma(-3/2)} n^{-5/2} 4^n,$$

which establishes part (ii) of Theorem 1.

Now we study the root face degree distribution of rooted simple circuit maps. The method is essentially the same as that used in [5]. Setting

$$z = Z\frac{Q(x,x,1)}{x^2(1+Y)},$$

and solving (1) for $Q(x,x,z)$, we obtain a messy expression for $\bar{C}(X,X,Z) = Q(x,x,z)$. For any positive constant ϵ and $|Z| \le 10/(9+\epsilon)$, it is not difficult to see that $\bar{C}(X,X,Z)$ is analytic in $\{X : |X| \le (1/4) + \delta, X \ne 1/4\}$ for some small positive δ. With the help of Maple, we obtain the following asymptotic expansion as $X \to 1/4$:

$$\bar{C}(X,X,Z) = b_0(Z) + b_1(Z)(1-4X) + b_2(Z)(1-4X)^{3/2} + O\left((1-4X)^2\right), \qquad (12)$$

where

$$b_2(Z) = \frac{Z(27Z^3 - 315Z^2 + 800Z - 500 + (5-Z)(10-9Z)\sqrt{(10-Z)(10-9Z)})}{25(10-9Z)\sqrt{(10-Z)(10-9Z)}},$$

and the big-O is uniform in the closed disk $|Z| \leq 10/(9+\epsilon)$ for any positive constant ϵ. It follows from Flajolet-Odlyzko's transfer theorem [10] that

$$[X^n]\bar{C}(X,X,Z) \sim \frac{3b_2(Z)}{4\sqrt{\pi}} n^{-5/2} 4^n, \tag{13}$$

uniformly for $|Z| \leq 10/(9+\epsilon)$ as $n \to \infty$. Hence, for each fixed k,

$$q_k = \lim_{n\to\infty} \frac{\bar{C}_{n,k}}{\bar{C}_n} = \frac{25}{8}[Z^k]b_2(Z).$$

Applying Darboux's lemma again, we obtain

$$q_k \sim \frac{25}{162}\sqrt{\frac{k}{2\pi}}(9/10)^k, \text{ as } k \to \infty.$$

This establishes part (iii) of Theorem 1.

Now we consider arbitrary k. Using (13), we obtain

$$[X^n]\bar{C}(X,X,Z) = O\left(\bar{C}_n\right),$$

uniformly for $|Z| \leq 10/(9+\epsilon)$ as $n \to \infty$. Thus

$$\bar{C}_{n,k}Z^k \leq [X^n]\bar{C}(X,X,Z) = O\left(\bar{C}_n\right)$$

for $Z = 10/(9+\epsilon)$ and any integer $k \geq 0$. This gives part (iv) of Theorem 1. ∎

4 Remarks

By Theorem 1, the average root face degree of a random simple circuit map is bounded. However, it can be shown that the average root face degree of a random circuit map is not bounded. In fact, one may use (1), (4), (8), and (9) to derive

$$\left.\frac{\partial C(X,X,Z)}{\partial Z}\right|_{Z=1} = K(1 - X/X_0)^{-1/2} + O(1),$$

as $X \to X_0$ for some positive constant K. Hence the average root face degree of a random n-edge circuit map is asymptotic to cn for some positive constant c.

It is also interesting to note that the generating functions for rooted k-connected planar maps, $k = 1, 2, 3$, and for rooted simple circuit maps, all admit a rational parametrization:

$$F(x) = \phi(t), \ x = \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are both rational functions of t. Such a rational parametrization is usually the first step towards getting a closed form expression for the coefficients by using Lagrange inversion. However, the generating function $C(X,X,1)$ for rooted circuit maps does not admit such a rational parametrization. This claim can be proved, using Maple, in the following two steps.

Step 1 Use (10) and (11) to eliminate the parameter u and to obtain a polynomial equation in X and $C(X,X,1)$.

Step 2 Using the above polynomial equation, one finds that the genus of $C(X, X, 1)$ is equal to 2. Since an algebraic function has a rational parametrization if and only if it has genus 0 (see, e.g., [1]), $C(X, X, 1)$ does not have a rational parametrization.

One may also apply the method of [6] to show that almost all n-edge rooted simple circuit maps contain cn copies of any given 3-c planar map for some positive constant c. Hence it follows from [18] that almost all rooted simple circuit maps are asymmetric. This implies that the number of (unrooted) n-edge circuit maps (G, C) with $|C| = k$ is asymptotically $\bar{C}_{n,k}/(2k)$. Summing over k and using parts (iii) and (iv) of Theorem 1, we see that the number of unrooted n-edge circuit maps (G, C) is asymptotic to $cn^{-5/2}4^n$ for some positive constant c.

Finally, it easily follows from Whitney's theorem [20] that a circuit graph (G, C) has a unique embedding in the plane so that C bounds a face. Hence the number of circuit graphs (G, C) is the same as the number of unrooted circuit maps.

It is also interesting to compare $\bar{C}_n$ with the number of rooted 2-connected simple maps. The following result may not be new; however, we are unable to find it in the literature, so we sketch its proof here for self completeness.

Lemma 1 *The number of rooted 2-connected n-edge simple maps is asymptotic to*

$$\frac{352}{675}\sqrt{\frac{1}{15\pi}}n^{-5/2}(729/128)^n.$$

Proof: Let $B(x)$ be the generating function for rooted 2-connected maps, and $\bar{B}(x)$ for rooted 2-connected simple maps. It is convenient to treat the link map (two distinct vertices joined by an edge) as 2-connected. By Proposition 1, we have

$$B(x) = Q(x, x, 1) + x = u(1 - u - u^2),\ x = u(1 - u)^2.$$

Since each rooted 2-connected map can be obtained from a rooted 2-connected simple map by replacing some edges with a rooted 2-connected map whose root face degree is 2, we have $B(x) = \bar{B}(x(1 + B(x)))$. This gives the following parametric expression for $\bar{B}(X)$:

$$\bar{B}(X) = u(1 - u - u^2),\ X = u(1 - u)^3(1 + u)^2.$$

Hence u and $\bar{B}$ are both algebraic functions of X. The dominant singularity of $u(X)$ and $\bar{B}(X)$ is obtained by solving

$$X = u(1 - u)^3(1 + u)^2,\ X'(u) = 0,$$

which gives $u = 1/3$ and $X = 128/729$. Also $\bar{B}(X)$ has the following asymptotic expansion at $X = 128/729$:

$$\bar{B}(X) = \frac{5}{27} - \frac{32}{135}\left(1 - \frac{729}{128}X\right) + \frac{1408\sqrt{15}}{30375}\left(1 - \frac{729}{128}X\right)^{3/2} + \cdots.$$

Now the lemma follows immediately from Darboux's lemma. ∎

The following table summarizes the asymptotic numbers for 2-c maps, circuit maps, 3-c maps, 2-c simple maps, and simple circuit maps.

type of map	allow multiple edges	no multiple edges
2-connected map	$\frac{2}{9}\sqrt{\frac{1}{3\pi}}n^{-5/2}(27/4)^n$	$\frac{352}{675}\sqrt{\frac{1}{15\pi}}n^{-5/2}(729/128)^n$
circuit map	$\frac{0.147}{2}\sqrt{\frac{1}{\pi}}n^{-3/2}4.08^n$	$\frac{6}{25}\sqrt{\frac{1}{\pi}}n^{-5/2}4^n$
3-connected map	$\frac{2}{243}\sqrt{\frac{1}{\pi}}n^{-5/2}4^n$	$\frac{2}{243}\sqrt{\frac{1}{\pi}}n^{-5/2}4^n$

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