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Irreducible Triangulations and Triangular Embeddings On a Surface

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ABSTRACT

A classical result of Steinitz says that there is only one irreducible triangulation on the sphere. D. Barnette showed that there are only two irreducible triangulations on the projective plane. Using Barnette's result, S. Lawrencenko showed that there are at most four different triangular embeddings of a 3-connected graph on the projective plane. In this paper, we give an explicit bound for the number of irreducible triangulations on any surface. We also show that, on any surface, the number of triangular embeddings of a 3-connected graph is bounded above by a constant depending only on the surface. Examples show that this does not hold for non-triangular embeddings of 3-connected graphs on the torus.

1. Introduction

A *map* M is a connected graph G embedded in a surface S (a compact 2-dimensional manifold) such that no edges cross each other and all components of $S - G$ are simply connected regions. These components are called *faces* of the map. G is called the *underlying graph* of M. A cycle of M is called *facial* if it bounds a face. Throughout the paper, all graphs have no loops or multiple edges. If all faces of a map M are triangles, then M is called a *triangulation* on S, or a *triangular embedding* of the graph G on S. A graph which has triangular embeddings is called a *triangular graph.* If v is a vertex of G, the *neighborhood graph* of v is the graph induced by v and its neighbors.

Let M be a triangular embedding of G. If an edge $\{v^+v^-\}$ lies in precisely two triangles, namely, the two facial triangles $\{v^+v^-u\}$ and $\{v^+v^-w\}$, then we can *shrink* $\{v^+v^-\}$ by contracting it into one vertex v while the two facial triangles $\{v^+v^-u\}$ and $\{v^+v^-w\}$ degenerate into edges $\{vu\}$ and $\{vw\}$. Clearly this operation yields another triangulation. Such an edge $\{v^+v^-\}$ is called a *shrinkable edge.* The inverse operation is *splitting* the vertex v along the edges $\{vu\}$ and $\{vw\}$. A triangulation without a shrinkable edge is called *irreducible.* By successively shrinking shrinkable edges, every triangulation can be reduced to one of the irreducible ones. Conversely, all triangulations can be generated from irreducible ones by successively splitting vertices.

Steinitz [6] showed that there is only one irreducible triangulation on the sphere, namely, the embedding of K_3; Barnette [2] showed that there are only two irreducible triangulations on the projective plane. Using Barnette's result, Lawrencenko [4] showed that there are at most four different triangular embeddings of a 3-connected graph on the projective plane. We prove that the Euler genus of an irreducible triangulation with a large number of vertices is large. This implies that, on any fixed surface, there are only a finite number of irreducible triangulations, and the number of triangular embeddings of a 3-connected graph is bounded above by a constant depending only on the surface.

Since the enumeration of maps can be regarded as counting graphs weighted according to the number of embeddings, we can, using the above results, translate properties of almost all triangulations to properties of almost all 3-connected graphs that have triangular embeddings. For example, Bender and Richmond [3] showed that, on any fixed surface S, an exponentially small fraction of triangulations with n edges are hamiltonian. It follows that an exponentially small fraction of the 3-connected graphs with n edges which have triangular embeddings on S are hamiltonian. Similarly almost no such graphs have a 2-factor and almost no such graphs with an even number of vertices have a 1-factor.

The Euler genus of a graph G is defined by $\bar{\gamma}(G) = \min\{2-\chi(S)\}$, where the minimum is taken over all surfaces on which G is embeddable, and $\chi(S)$ is the Euler characteristic of S. The following proposition is an immediate corollary from Archdeacon's results [1].

Proposition 1. *Let G_1 and G_2 be two subgraphs of a graph G. If G_1 and G_2 have at most two vertices in common, then $\bar{\gamma}(G) \geq \bar{\gamma}(G_1) + \bar{\gamma}(G_2)$.*

2. Theorems and proofs

Theorem 1. *Let G be a graph with n vertices, where every edge lies in at least three 3-cycles. Then $\bar{\gamma}(G) > n^{1/4}/12 - 3/2$.*

Proof: Since every edge lies in at least three 3-cycles, every vertex has degree at least four, and the neighborhood graph of any vertex is non-planar. We first show that, for fixed $\bar{\gamma}(G)$, G has many vertices of degree at most 6. Let n_i denote the number of vertices of degree i, and let e denote the number of edges. Using Euler's formula, we have

$$2 - \bar{\gamma}(G) \leq n - e/3,$$

which implies

$$e < 3(\bar{\gamma}(G) + n).$$

This together with the relations

$$\sum_{i \geq 4} i n_i = 2e, \quad \sum_{i \geq 4} n_i = n$$

gives

$$\sum_{i=4}^{6} n_i > n/3 - 2\bar{\gamma}(G),$$

i.e., G has more than $n/3 - 2\bar{\gamma}(G)$ vertices of degree ≤ 6.

Let A denote the set of pairwise non-adjacent vertices of degree ≤ 6, and let B denote the neighbors of A. Then $|A| > n/21 - 2\bar{\gamma}(G)/7$. We now consider the following cases:

1. Every vertex of B has at most $n^{3/4}/21$ neighbors in A. Then A has more than $n^{1/4}/6 - n^{-3/4}\bar{\gamma}(G)$ vertices whose neighborhood graphs are pairwise disjoint. By Proposition 1, we have $\bar{\gamma}(G) > n^{1/4}/6 - \bar{\gamma}(G)$, and hence $\bar{\gamma}(G) > n^{1/4}/12$.
2. B has a vertex x with more than $n^{3/4}/21$ neighbors in A. Let A' be the set of neighbors of x in A and let B' be the set of neighbors of A'. We now consider the following subcases:
   (1) Every vertex of $B' - \{x\}$ has at most $n^{1/2}/12$ neighbors in A'. As in case 1, there are more than $n^{1/4}/12$ vertices of A' whose neighborhood graphs have only one vertex x in common pair by pair. By Proposition 1, we have $\bar{\gamma}(G) > n^{1/4}/12$.
   (2) $B' - \{x\}$ has a vertex y with more than $n^{1/2}/12$ neighbors in A'. Let A'' be the set of neighbors of y in A' and let B'' be the set of neighbors of A''. If every vertex of $B'' - \{x, y\}$ has at most $n^{1/4}/6$ neighbors in A'', then there are more than $n^{1/4}/12$ vertices of A'' whose neighborhood graphs have only two vertices x and y in common pair by pair. By Proposition 1 again, we have $\bar{\gamma}(G) > n^{1/4}/12$. If $B'' - \{x, y\}$ has a vertex z with more than $n^{1/4}/6$ neighbors in A'', then G has a subgraph $K_{3,[n^{1/4}/6]}$. By Euler's formula, the bipartite graph $K_{3,s}$ has Euler genus

$$\geq 2 - ((3 + s) - 3s + 3s/2) \geq s/2 - 1.$$

   Hence $\bar{\gamma}(G) > n^{1/4}/12 - 3/2$. ∎

Corollary 1. *On any surface, there are only a finite number of irreducible triangulations.*

Proof: If S is the sphere, then by Steinitz's result, there is only one irreducible triangulation K_3 on S. If S is not the sphere and T is an irreducible triangulation on S, then

$\bar{\gamma}(T) \leq 2 - \chi(S)$ and every edge of T lies in at least three 3-cycles. By Theorem 1, the number of vertices of T is bounded above by a constant depending only on $\chi(S)$. Therefore, the number of irreducible triangulations on S is finite. ∎

Corollary 2. *The number of triangular embeddings of a 3-connected graph on a surface S is bounded above by a constant depending only on S.*

Proof: If a 3-connected graph G has a triangular embedding on S and it has a shrinkable edge e, then e lies in precisely two triangles and hence they must be facial in every triangular embedding. Therefore the natural map (shrinking e) from the triangular embeddings of G to the triangular embeddings of $G' = (G$ with e contracted and multiple edges deleted) is one to one. So G has no more triangular embeddings than G', and hence the number of triangular embeddings of G on S is no more than that of the underlying graphs of the irreducible triangulations on S. Our claim now follows from Corollary 1. ∎

Corollary 2 does not extend to non-triangular embeddings of 3-connected graph. Let G be a triangulation of the sphere, and let x_1 and x_2 be two non-adjacent vertices of G of degree d_1 and d_2, respectively. Let G' be obtained from G by adding the edge x_1x_2. We can embed G' on the torus as follows: Select a face F_i having x_i on its boundary, for $i = 1,\ 2$. Add a tube (handle) to the sphere from F_1 to F_2 and draw x_1x_2 on that. As F_i can be chosen in d_i ways, G has at least d_1d_2 embeddings on the torus. G and G' can even be chosen to be 5-connected. Any 6-connected graph on the torus is a triangulation and so Corollary 2 applies to any such graph. But, if we start with a triangulation G of the torus and construct G' as above, then we get a 6-connected graph which has many embeddings on the double-torus. For 7-connected graphs the problem becomes uninteresting, since, for any fixed surface S, there are only finitely many 7-connected graphs on S, by Euler's formula.

3. References

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