The Average Number of Non-Separating Edges in Catalan Triangulations of the Mőbius Band and the Cylinder

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Abstract

Let Σ be a surface whose boundary consists of a set of disjoint simple closed curves. A Catalan triangulation of Σ is a triangulation of Σ such that all vertices are on the boundary. An edge in a Catalan triangulation is called separating if the removal of its end vertices disconnects the triangulation. The number of Catalan triangulations of a disk is given by the well-known Catalan numbers. Catalan triangulations of the Mőbius band were counted recently by Edelman and Reiner. In this paper, we study the average number of separating edges in Catalan triangulations of the Mőbius band and the cylinder. The total number of Catalan triangulations of the cylinder is also given.

Keywords[1]: Catalan triangulation, Mőbius Band, Cylinder, Enumeration, Separating edge.

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1 Introduction and Definitions

In this paper, Σ denotes a surface whose boundary consists of a set of disjoint simple closed curves. We consider triangulations of Σ with all vertices on the boundary. Edelman and Reiner [5] called such triangulations *Catalan triangulations* since such triangulations of a disk are counted by the well known Catalan numbers. For enumeration purpose, a Catalan triangulation is *rooted* by specifying a vertex (called the *root vertex*), and an edge on the boundary incident with the root vertex (called the *root edge*).

Enumeration of Catalan triangulations of a disk can be traced back to Euler [3]. Enumeration of general planar triangulations (and maps) were initiated by Tutte [9, 10] in 1960s, aiming at the famous four color problem. Major developments on Enumeration of maps on general surfaces occurred in 1980s mainly because of the connections with quantum physics [2, 7], and other branches of mathematics [8]. We note that enumerative map theory usually treats triangulations (and maps in general) on compact surfaces without boundary. Catalan triangulations treated in this paper are closely related to near-triangulations on surfaces without boundary, by the simple operation of capping each simple closed curve on the boundary with a disk. For example, given a Catalan triangulation on the Mőbius band, we can cap the boundary of the Mőbius band with a disk to obtain a rooted near-triangulation on the projective plane, with all the vertices on the root face.

In the following, all Catalan triangulations shall be rooted, and have no loops or multiple edges. An edge in a Catalan triangulation is *separating* if the removal of its end vertices disconnects the triangulation. An edge is called *non-separating* if it is not on the boundary and is not separating.

Let D_n and M_n be the number of Catalan triangulations with n vertices, of a disk and Mőbius band, respectively. , and it is well-known that

$$D_n = \frac{1}{n-1}\binom{2n-4}{n-2},$$

and its generating function is given by

$$D(x) = \sum_{n\geq 3} D_n x^n = \frac{x(1-\sqrt{1-4x})}{2} - x^2 \tag{1}$$

Recently Edelman and Reiner [5] obtained the following expression for the generating function $M(x) = \sum M_n x^n$:

$$\begin{aligned} M(x) &= \frac{2-13\,x+20\,x^2+5\,x^3-12\,x^4}{2x^2(1-4\,x)} - \frac{2-9\,x+6\,x^2+7\,x^3-2\,x^4}{2x^2\sqrt{1-4\,x}} \\ &= x^5+14x^6+113x^7+720x^8+\cdots. \end{aligned} \tag{2}$$

In this paper, we study the average number of separating edges in Catalan triangulations of the Mőbius band and the cylinder. Our main results are as follows.

Theorem 1 *The average number of non-separating edges in a Catalan triangulation of the Mőbius band with n vertices is given by*

$$\sqrt{\frac{4n}{\pi}} + \frac{29}{2\pi} + O\left(\frac{1}{\sqrt{n}}\right)$$

Theorem 2 **(i)** *Let C_n be the number of Catalan triangulations of a cylinder with n vertices. Then*

$$\begin{aligned}\sum C_n x^n &= \frac{2-13\,x+21\,x^2-2\,x^3+x^4-4\,x^5}{2x^2\,(1-4\,x)} - \frac{2-9\,x+7\,x^2+2\,x^3+3\,x^4}{2x^2\sqrt{1-4\,x}}\\ &= 7x^6+77x^7+555x^8+3318x^9+17861x^{10}+89980x^{11}+433292x^{12}+\cdots\end{aligned}$$

(ii) *The average number of nonseparating edges in a Catalan triangulation of n vertices of a cylinder is*

$$\sqrt{\frac{4n}{\pi}}+\frac{59}{4\pi}+O\left(\frac{1}{\sqrt{n}}\right).$$

There are heavy algebraic manipulations involved in this paper, which are handled by the computer algebra system *Maple.*

2 Catalan triangulations of a disk

To enumerate Catalan triangulations of the Mőbius band and cylinder, we need to consider three special types of Catalan triangulations of a disk, which shall be called J-type, K-type and L-type. Let S be the set of Catalan triangulations of a disk containing a distinguished vertex v_1'' such that v_1'' is distinct from the root vertex v_1' and there is no edge on the boundary of the disk joining v_1' and v_1''. There are two paths on the boundary of the disk between vertices v_1' and v_1'', the one containing the root edge $v_1'v_3$ shall be called the right path and the other one the left path (See Figure 1). A member of S is of J-type if there is no path of length less than 3 between v_1 and v_1''. Let T be a J-type triangulation, let $v_1'v_2'$ and $v_1''v_2''$ be the edges on the left path between v_1' and v_1''. T is called of K-type if there is no path of length less than 3 between v_2' and v_2''. T is called of L-type if there is no edge joining v_2'' and v_3.

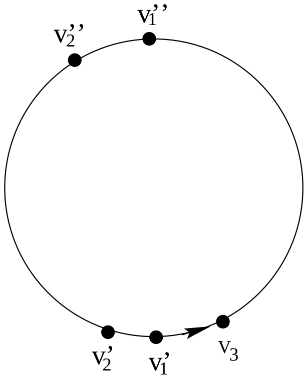


Figure 1: Special types of Catalan triangulations of a disk

Let $S_{i,j}$, $J_{i,j}$, $K_{i,j}$ and $L_{i,j}$ be the number of Catalan triangulations in S, of J-type, of K-type and of L-type, respectively, where $i+2$ is the number of vertices on the left path between v_1' and v_1'', and j is the number of vertices on the right path between v_1' and v_1''. Define generating functions

$$S(u,v) = \sum_{i,j} S_{i,j}u^iv^j = \sum_i S_i(v)u^i,$$

$$\begin{aligned}
J(u,v) &= \sum_{i,j} J_{i,j} u^i v^j = \sum_i J_i(v) u^i,\\
K(u,v) &= \sum_{i,j} K_{i,j} u^i v^j = \sum_i K_i(v) u^i,\\
L(u,v) &= \sum_{i,j} L_{i,j} u^i v^j = \sum_i L_i(v) u^i,
\end{aligned}$$

We first prove

Lemma 1

$$J(x,x) = \frac{1-7x+14x^2-7x^3}{x\sqrt{1-4\,x}} - \frac{1-5x+6\,x^2-x^3}{x} \tag{3}$$

$$J_2(x) = \frac{1-4x+x^2+2x^3}{2x} - \frac{(1-2x-x^2)\sqrt{1-4x}}{2x} \tag{4}$$

$$K(x,x) = \frac{2-15x+32x^2-14x^3-2x^4-8x^5}{x\sqrt{1-4x}} - \frac{2-11x+14x^2+2x^4-2x^5}{x} \tag{5}$$

$$\begin{aligned}
\frac{\partial K}{\partial u}(x,x) &= \frac{1-4x-7x^2+28x^3-3x^4+16x^5-16x^6}{x(1-4x)}\\
&\quad - \frac{1-6x-x^2+46x^3-39x^4+6x^5-56x^6}{x^2(1-4x)^{3/2}}
\end{aligned} \tag{6}$$

$$L(x,x) = \frac{2-13x+20x^2+5x^3-12x^4}{x\sqrt{1-4x}} - \frac{2-9x+6x^2+7x^3-2x^4}{2x} \tag{7}$$

Proof: We first prove (3) using the principle of inclusion and exclusion. We consider the following three conditions for Catalan triangulations in S.

Condition J_1 There is an edge joining v_1' and v_1''.

Condition J_2 There is a path $v_1' a v_1''$ of length two such that a is on the left path between v_1' and v_1''.

Condition J_3 There is a path $v_1' a v_1''$ of length two such that a is on the right path between v_1' and v_1''.

We have

$$S(u,v) = \sum_{n\geq 4} D_n \sum_{i=1}^{n-3} u^i v^{n-i} = \frac{u}{v-u}\left(D(v) - \frac{v^3}{u^3} D(u)\right). \tag{8}$$

In the following, we use $N(J_1)$ to denote the generating function of the Catalan triangulations in S satisfying condition J_1, and so on. Those generating functions will be easily computed using decompositions described by the corresponding pictures, so we leave the details to the reader. For example, the triangulations satisfying condition J_1 are decomposed into two Catalan triangulations of a disk (See Figure 2(J_1)). Hence we have

$$N(J_1) = u^{-2} D(u) D(v)$$

Similarly, the generating functions of the Catalan triangulations in S satisfying conditions J_2 and J_3, respectively, are given by

$$\begin{aligned}
N(J_2) &= u^{-3} v^{-1} \left(D(u)+u^2\right)^2 \left(D(v)-v^3\right)\\
N(J_3) &= u^{-3} v^{-1} \left(D(u)-u^3\right) \left(D(v)+v^2\right)^2
\end{aligned}$$

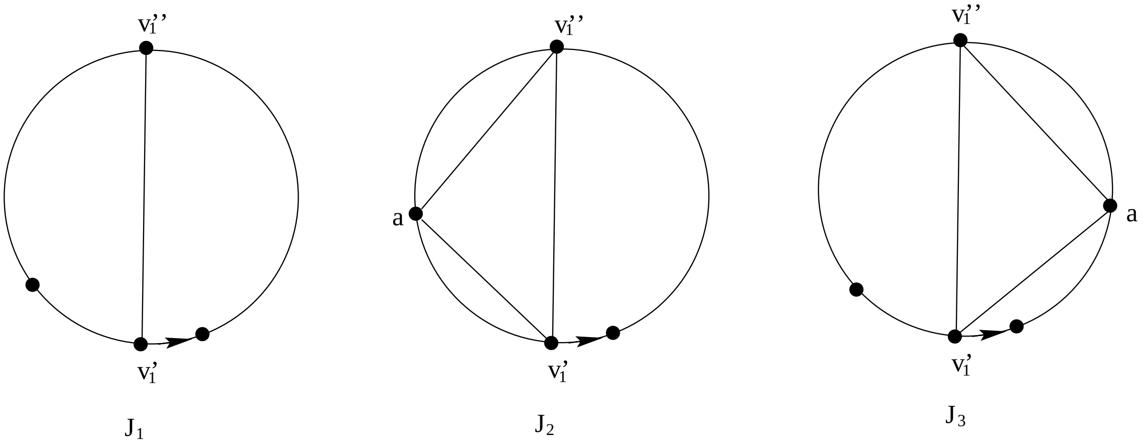


Figure 2: Counting J-type Catalan triangulations

The generating functions of the Catalan triangulations in S satisfying two of the conditions are given by

$$\begin{aligned} N(J_1J_2) &= u^{-3}\left(D(u)+u^2\right)^2 D(v) \\ N(J_1J_3) &= u^{-2}v^{-1}D(u)\left(D(v)+v^2\right)^2 \\ N(J_2J_3) &= 2u^{-3}v^{-1}\left(D(u)+u^2\right)^2\left(D(v)+v^2\right)^2 \end{aligned}$$

The generating function of the Catalan triangulations in S satisfying all the three conditions is given by

$$N(J_1J_2J_3) = u^{-3}v^{-1}\left(D(u)+u^2\right)^2\left(D(v)+v^2\right)^2.$$

By the principle of inclusion and exclusion, we obtain

$$J(u,v) = S(u,v) - N(J_1) - N(J_2) - N(J_3) + N(J_1J_2) + N(J_1J_3) + N(J_2J_3) - N(J_1J_2J_3). \quad (9)$$

Expressions (3) and (4) follow from (1) and some algebra, with the help of Maple.

Expression (5) for $K(u,v)$ follows from a similar argument used above by using the generating function $J(u,v)$ for the ground set and the following three conditions :

Condition K_1 There is an edge joining v_2' and v_2''.

Condition K_2 There is a path $v_2'av_2''$ of length two such that a is on the left path between v_2' and v_2''.

Condition K_3 There is a path $v_2'av_2''$ of length two such that a is on the right path between v_2' and v_2''.

We have (See Figure 3)

$$\begin{aligned} N(K_1) &= \left(D(u)+u^2\right) J_2(v) \\ N(K_2) &= u^{-1}\left(D(u)+u^2\right)^2 J_3(v) \\ N(K_3) &= u^{-1}D(u)\left(D^2(v)v^{-3} - \left(D(v)+v^2\right)^2 v^{-1}\right) \end{aligned}$$

$$\begin{aligned}
N(K_1K_2) &= u^{-1}\left(D(u)+u^2\right)^2 J_2(v)\\
N(K_1K_3) &= \left(D(u)+u^2\right)^2\left(D^2(v)v^{-3}-\left(D(v)+v^2\right)^2 v^{-1}\right)\\
N(K_2K_3) &= 2u^{-1}\left(D(u)+u^2\right)^2\left(D^2(v)v^{-3}-\left(D(v)+v^2\right)^2 v^{-1}\right)\\
N(K_1K_2K_3) &= N(K_2K_3)/2.
\end{aligned}$$

Now (5) and (6) follow from (1) and the formula

$$\begin{aligned}
K(u,v) &= J(u,v)-u^2[u^2]J(u,v)-N(K_1)-N(K_2)-N(K_3)\\
&\quad +N(K_1K_2)+N(K_1K_3)+N(K_2K_3)-N(K_1K_2K_3). \qquad (10)
\end{aligned}$$

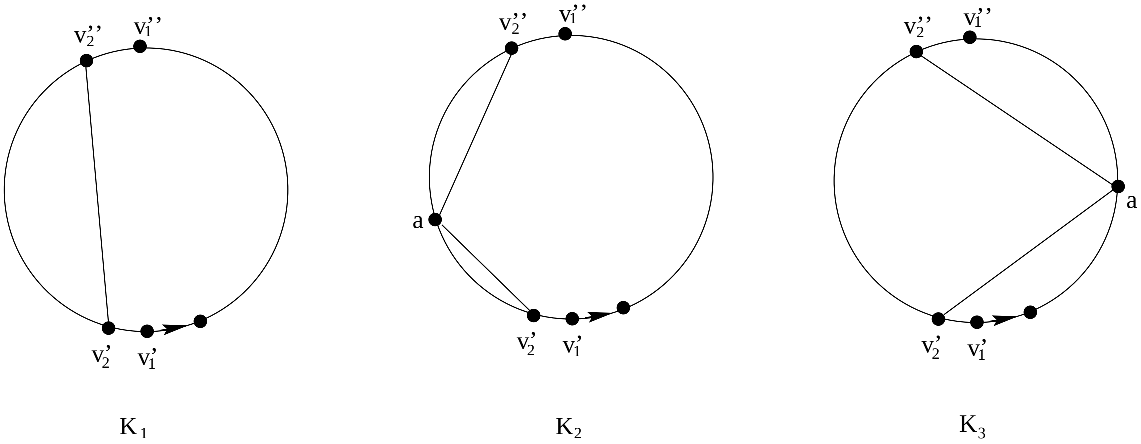


Figure 3: Counting K-type Catalan triangulations

Now we prove (7). Let $Q(u,v)$ be the generating function for those J-type Catalan triangulations containing an edge $v_2''v_3$. It is easy to see that

$$Q(u,v)=u^{-2}\left(D(u)-uD(u)-u^3\right)\left(D(v)-vD(v)-v^3\right).$$

Using $L(u,v)=J(u,v)-Q(u,v)$, we obtain (7). ∎

3 Catalan triangulations of the Mőbius band

The expression for $M(x)$ follows quite easily from (7), we include a short proof here for self completeness.

Let v_2v_3 be the root edge of a Catalan triangulation T of the Mőbius band, and let v_1 be the third vertex on the triangular face containing v_2v_3. We delete v_2v_3 and cut through v_1. If the decomposition gives two disjoint pieces, one being a Catalan triangulation of a disk and the other being a Catalan triangulation of the Mőbius band (See Figure 4), we obtain the contribution

$$2x^{-1}(D(x)+x^2)M(x).$$

If the decomposition gives a single piece, then it is a Catalan triangulation of L-type (See Figure 5), and we obtain the contribution $x^{-1}L(x,x)$. Hence we have

$$M(x) = 2x^{-1}(D(x) + x^2)M(x) + x^{-1}L(x,x).$$

Solving for $M(x)$ and using (1) and (7), we obtain (2).

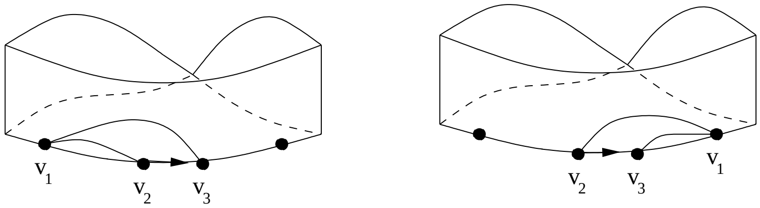
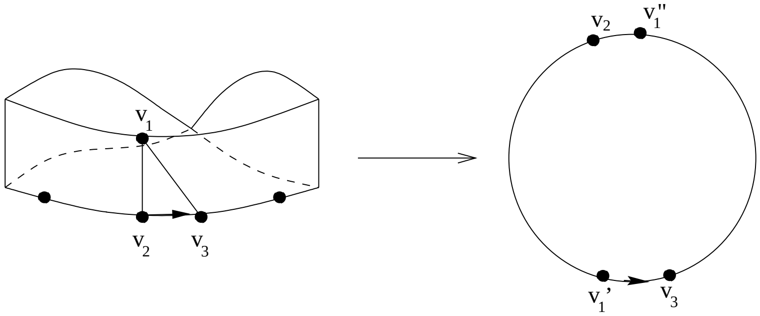


Figure 4: The decomposition gives two disjoint pieces



(a)

(b)

Figure 5: The decomposition gives a Catalan triangulation of L-type

For enumeration purpose, it seems easier to deal with separating edges than nonseparating edges in Catalan triangulations of the Mőbius band. Let $\bar{M}_n$ be the number of Catalan triangulations of the Mőbius band with n vertices and with a distinguished separating edge. Then $\bar{M}_n/M_n$ gives the average number of separating edges in a Catalan triangulation of the Mőbius band with n vertices. Using Euler's formula, it is easy to see that the total number of edges in such a triangulation is $2n$ which includes n edges on the boundary of the Mőbius band. Hence the average number of nonseparating edges in a Catalan triangulation of the Mőbius band with n vertices is given by

$$E_n = n - \bar{M}_n/M_n \tag{11}$$

Let $\bar{M}(x) = \sum \bar{M}_n x^n$, and $X = (1-4x)^{1/2}$. We first show

Lemma 2

$$\begin{aligned}\bar{M}(x) \;=\; & \Big(8 - 13X - 95X^2 + 384X^3 - 592X^4 + 430X^5 - 102X^6 - 40X^7 \\ & +16X^8 + 7X^9 - 3X^{10}\Big) \Big/ \Big(32X^4(1+X)^3\Big)\end{aligned}$$

Proof: Let T be a Catalan triangulation of the Mőbius band with a distinguished edge d. Cutting through the distinguished edge d creates a Catalan triangulation T_1 of a disk and a Catalan triangulation T_2 of the Mőbius band. There are two cases.

Case 1 The root edge of T is contained in T_1. In this case, T_1 contains a copy of d on its boundary which is distinct from the root edge of T_1; a copy of d in T_2 serves as the root edge of T_2. The generating function corresponding to this case is given by

$$x^{-2}M(x)\sum_{n\geq 3}(n-1)D_n x^n = x^{-2}M(x)\left(xD'(x)-S(x)\right) \tag{12}$$

Case 2 The root edge of T is contained in T_2. In this case, a copy of d serves as the root edge of T_1; T_2 contains a copy of d on its boundary which is distinct from the root edge of T_2. The generating function corresponding to this case is given by

$$x^{-2}D(x)\sum_{n\geq 3}(n-1)M_n x^n = x^{-2}D(x)\left(xM'(x)-M(x)\right) \tag{13}$$

Hence we have

$$\bar{M}(x) = x^{-2}M(x)\left(xD'(x)-D(x)\right) + x^{-2}D(x)\left(xM'(x)-M(x)\right).$$

Now Lemma 2 follows from (1) and (2), using $x = (1-X^2)/4$. ∎

Now we are ready to prove Theorem 1. We first expand $M(x)$ and $\bar{M}(x)$ as power series of X:

$$\begin{aligned} M(x) &= \frac{1}{4}X^{-2} - \frac{29}{16}X^{-1} + \frac{49}{8} + O\left(X\right), \\ \bar{M}(x) &= \frac{1}{4}X^{-4} - \frac{37}{32}X^{-3} - \frac{1}{4}X^{-2} + O\left(X^{-1}\right). \end{aligned}$$

Now it follows from [4, (2.2)] that

$$M_n = 4^{n-1} - \frac{29}{4\sqrt{\pi}}n^{-1/2}4^{n-1} + O\left(n^{-3/2}4^n\right), \tag{14}$$

$$\bar{M}_n = n4^{n-1} - \frac{37}{4\sqrt{\pi}}n^{1/2}4^{n-1} + O\left(n^{-1/2}4^n\right). \tag{15}$$

Hence

$$\frac{\bar{M}_n}{M_n} = n - \sqrt{\frac{4n}{\pi}} - \frac{29}{2\pi} + O\left(\frac{1}{\sqrt{n}}\right).$$

This together with (11) gives Theorem 1.

4 Catalan triangulations of a cylinder

We can think of a cylinder as a planar region between two concentric circles. We use the convention that the root edge of a Catalan triangulation of a cylinder is on the exterior circle. Let $C_{i,j}$ be the number of Catalan triangulations of a cylinder, having i vertices on the exterior circle and j vertices on the interior circle. Define $\bar{C}_{i,j}$ similarly for Catalan triangulations of a cylinder containing a distinguished nonseparating edge. Define the generating functions

$$\begin{aligned} C(u,v) &= \sum_{i\geq 3, j\geq 3} C_{i,j}u^i v^j \\ \bar{C}(u,v) &= \sum_{i\geq 3, j\geq 3} \bar{C}_{i,j}u^i v^j \end{aligned}$$

We first establish the following lemma.

Lemma 3

$$\begin{aligned} C(u,v) &= \frac{v^{-1}\left(J(u,v)-\left(D(u)+u^2\right)J_2(v)\right)}{1-2u^{-1}(D(u)+u^2)}, && (16)\\ \bar{C}(u,v) &= u^{-1}\frac{\partial K(u,v)}{\partial v}-(uv)^{-1}K(u,v). && (17)\end{aligned}$$

Proof: Let T be a Catalan triangulation of a cylinder with root edge v_2v_3 , and let v_1 be the third vertex on the triangular face containing v_2v_3. If v_1 is on the exterior circle, then removing v_2v_3 and cutting through v_1 gives a Catalan triangulation of a disk and a Catalan triangulation of a cylinder. So the contribution in this case is

$$2u^{-1}\left(D(u)+u^2\right)C(u,v)$$

If v_1 is on the interior circle, then removing v_2v_3 and cutting through v_1 gives a Catalan triangulation of a disk of J-type such that v_2 and v_3 are not joined by an edge. So the contribution in this case is

$$v^{-1}\left(J(u,v)-\left(D(u)+u^2\right)J_2(v)\right).$$

Hence we have

$$C(u,v)=2u^{-1}(D(u)+u^2)C(u,v)+v^{-1}\left(J(u,v)-uJ_1(v)-\left(D(u)+u^2\right)J_2(v)\right).$$

Solving for $C(u,v)$ gives (16).

For any Catalan triangulation T of a cylinder containing a distinguished nonseparating edge v_1v_2, cutting through v_1v_2 gives a Catalan triangulation of K-type. We can treat $v_1'v_2'$ as the root edge and the original root edge e of T as a distinguished edge. Noting that e must lie on the right path between v_1' and v_1'', we obtain (17). ∎

Proof of Theorem 2 : Using Lemma 3, $C(x,x)=\sum C_nx^n$, and Maple, we obtain Theorem 2(i), and

$$\begin{aligned}\bar{C}(x,x) \;=\; & \frac{5-46\,x+136\,x^2-144\,x^3+65\,x^4+8\,x^5-96\,x^6+160\,x^7-516\,x^8+528\,x^9}{x^3(1-4\,x)^{3/2}} - \\ & \frac{5-36\,x+74\,x^2-48\,x^3+23\,x^4+42\,x^5-40\,x^6+128\,x^7-292\,x^8+144\,x^9}{x^3(1-4\,x)} \qquad (18)\end{aligned}$$

Expanding $C(x,x)$ and $\bar{C}(x,x)$ as power series in $X=(1-4x)^{1/2}$, we obtain

$$\begin{aligned} C(x,x) &= \frac{1}{4}X^{-2}-\frac{59}{32}X^{-1}+\frac{201}{32}+O(X),\\ \bar{C}(x,x) &= \frac{1}{4}X^{-3}+O\left(X^{-1}\right).\end{aligned}$$

If we use $\bar{C}_n$ to denote the coefficient of x^n in the power series expansion of $\bar{C}(x,x)$, then, using [4, (2.2)], we obtain

$$\begin{aligned} C_n &= 4^{n-1}\left(1-\frac{59}{8\sqrt{n\pi}}+O\left(n^{-3/2}\right)\right),\\ \bar{C}_n &= \sqrt{\frac{4n}{\pi}}4^{n-1}\left(1+O(1/n)\right).\end{aligned}$$

Hence the average number of non-separating edges is $C_n/\bar{C}_n$, which gives Theorem 2(ii).

5 Remarks and open problems

There are considerable amount of research on map enumerations, done by both physicists and combinatorialists (See survey papers [1, 12]). However, exact enumeration of nonplanar maps seems to be quite difficult, especially when loops and multiple edges are not allowed. This paper deals with exact enumeration of Catalan triangulations, which have all the vertices on the boundary. For general triangulations with interior vertices, the enumeration problem is much more involved. Exact enumeration of general triangulations on the projective plane, which allows multiple edges but not loops, is done in [6]. Exact enumeration of triangulations on the projective plane with no multiple edges or loops is done in [11], which will appear in a forthcoming paper. The next natural step would be to derive exact number of triangulations on the torus and the Klein bottle. Catalan triangulations of the punctured torus has also been enumerated in [11], and the generating function is

$$\begin{aligned} T(x) &= \frac{40-560x+2900x^2-6512x^3+4936x^4+1664x^5-1640x^6+176x^7-960x^8}{x^4(1-4x)^2(1+\sqrt{1-4x})^2} \\ &\quad -\frac{4\sqrt{1-4x}(10-120x+505x^2-818x^3+228x^4+336x^5-24x^6+100x^7-48x^8)}{x^4(1-4x)^2(1+\sqrt{1-4x})^2} \\ &= 2x^6+68x^7+1070x^8+11060x^9+89740x^{10}+\cdots \end{aligned} \tag{19}$$

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