On Alon-Saks-Seymour Conjecture

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Abstract

A famous result of Graham and Pollak states that the complete graph with n vertices can be edge partitioned into $n-1$, but no fewer, complete bipartite graphs. In a quest for a combinatorial proof of this theorem, Alon, Saks and Seymour conjectured that if a graph can be edge partitioned into n complete bipartite graphs, then it is $(n+1)$-colourable. Rho verified the truth of the conjecture for $n \leq 4$. In this paper, we introduce a combinatorial approach to this conjecture using extended cubes and 2-arc-labeled tournaments. We also introduce reformulations of this conjecture and we show that the Alon-Saks-Seymour Conjecture is true for $n \leq 9$. Furthermore a related problem of edge partitioning a graph into induced bipartite graphs is discussed.

Key words: bipartite partition, chromatic number, squashed cube, tournament

1 Introduction

Let G be a graph. We define the *complete bipartite edge-partition number*, $CB(G)$, of G to be the minimum number of edge disjoint complete bipartite subgraphs of G whose union is G. Note that such a partition always exists as each edge can be considered as a complete bipartite graph by itself.

An *induced bipartite subgraph* of a graph G is a bipartite subgraph B with bipartition (X, Y) such that if $\{x, y\}$, $x \in X$ and $y \in Y$, is an edge of G then it is also an edge of B. Note that an induced bipartite subgraph is not necessarily an induced subgraph as G may have an edge inside X or inside Y. We define the *induced bipartite edge–partition number* to be analogous to $CB(G)$ and denote it by $IB(G)$. Note that we always have $IB(G) \leq CB(G)$.

We are interested in the following two questions.

Problem 1 *What is the maximum possible chromatic number of a graph with $CB(G) \leq k$?*

Problem 2 *What is the maximum possible chromatic number of a graph with $IB(G) \leq k$?*

It is a famous result of Graham and Pollak [3,4] that a complete graph of order n cannot be edge partitioned into less than $n-1$ complete (or induced) bipartite graphs. This simple combinatorial statement has only been proved using linear algebra and eigenvalue techniques. Thus it has been an intriguing question to find a combinatorial proof of it. This quest has given rise to several extensions of the theorem, some with proofs (such as [2]) and some as conjecture. Of those the following one has received most attention not only because of the Graham-Pollak theorem but also because of its similarity to the Erdős-Faber-Lovász conjecture.

The Alon–Saks–Seymour Conjecture [5,6]. If the edges of a graph G can be partitioned into k complete bipartite graphs, then G is $k+1$ colourable.

In this paper we introduce a combinatorial approach to this conjecture which provides a combinatorial proof of the conjecture avoiding case analysis for $n \leq 5$ and also proves the result for $n \leq 9$. Using graph homomorphisms we characterize all the critical graphs that one need to check in order to prove the conjecture for a fixed value of n. We show how one can encode these large graphs in smaller graphs. While we note that the conjecture is now mostly believed to be false, by proving it for $n \leq 9$, we show that a counterexample, if it exists, can only be found for large values of n. We also introduce some concrete suggestions on where one should look for such a counterexample.

2 Extended cubes and homomorphisms

Here we reduce Problems 1 and 2 to homomorphism problems. Let $\mathcal{M} = (\mathbb{Z}_2 \cup \{\infty\}, +, 0)$ be a monoid in which addition of elements in $\mathbb{Z}_2$ is done as in $\mathbb{Z}_2$ and $x + \infty = \infty$ for every x. Let $\mathcal{B}$ be the set of all elements of $\mathcal{M}^n$ that have a single 1 in their coordinates ($\mathcal{B}$ has $n2^{n-1}$ elements). We define $EC(n)$ (the *extended cube* of dimension n) to be the graph whose vertex set is $\mathcal{M}^n - \vec{\infty}$, $\vec{\infty} = (\infty, \infty, \ldots, \infty)$, with two vertices x and y being adjacent if and only if $x + y \in \mathcal{B}$. Note that the hypercube and squashed cubes [3,9] of dimension n are induced subgraphs of $EC(n)$ (with the property that any two vertices never disagree in positions where both have elements that are not ∞). We will use the term *vertex* when referring to undirected graphs and reserve the term *node* for discussion of directed tournaments later in the paper.

One can extend the definition of Hamming distance on hypercubes to $\mathcal{M}^n$ by assuming that the distance between ∞ and anything else is 0. This definition is convenient although (as it is not actually a metric) it abuses the usual definition of a distance function. It was introduced by Graham and Pollak [3] for its important application to the isometric embedding of graphs in squashed cubes (as a next best possibility for graphs not embeddable on a hypercube).

With this notation, $EC(n)$ is a graph on $\mathcal{M}^n - \vec{\infty}$ with x and y being adjacent if they are at Hamming distance exactly 1. A subgraph of $EC(n)$ in which every pair of vertices are at Hamming distance at most k is called a *k-ball* subgraph. The graph $EC(n)$ has $3^n - 1$ vertices, it is connected and, moreover, it is of diameter 2 as the following proposition shows.

Proposition 3 *The graph $EC(n)$ is of diameter 2.*

Proof: Let x and y be two distinct vertices of $EC(n)$. If x and y are not adjacent then we can find coordinates i and j such that neither x_i nor y_j is ∞, i.e., they are 0 or 1. Now let a be the vertex with $a_i = 1 + x_i$ and $a_k = \infty$ for $k \neq i$. Let b the vertex with $b_j = 1 + y_j$ and $b_k = \infty$ for $k \neq j$. Note that a is automatically adjacent to x and b is adjacent to y, if a or b is adjacent to both, then we are done, otherwise the vertex c defined by $c_i = 1 + x_i$, $c_j = 1 + y_j$ and $c_k = \infty$ for $k \neq i, j$ will be adjacent to both. □

The next two lemmas show the first connection between $EC(n)$ and edge partitions into certain bipartite subgraphs.

Lemma 4 *The edge set of $EC(n)$ can be partitioned into n induced bipartite subgraphs.*

Proof: Let B_i be the bipartite subgraph with bipartition (X_i, Y_i), where X_i consists of all vertices $x \in \mathcal{M}^n - \vec{\infty}$, with $x_i = 0$ and Y_i consists of all vertices $y \in \mathcal{M}^n - \vec{\infty}$, with $y_i = 1$. Each edge $\{x, y\}$ belongs to a unique B_i by the definition of adjacency. Thus $\{B_i\}_{i=1}^n$ partitions $E(EC(n))$ into n induced bipartite subgraphs. □

Lemma 5 *Each induced 1-ball subgraph of $EC(n)$ has complete bipartite edge partitioning number at most n.*

Proof: Let M be a 1-ball subgraph of $EC(n)$. Then for each i, where $1 \leq i \leq n$, we define B_i to be a bipartite graph where X_i is the set of vertices in $V(M)$ with 0 in coordinate i and Y_i being the set of vertices in $V(M)$ with 1 in their ith coordinate. As before each edge of M belongs to a unique B_i, however the absence of any pair of vertices at Hamming distance 2 or more ensures that B_i is a complete bipartite subgraph. Hence $\{B_i\}_{i=1}^n$ is a set of edge-disjoint complete bipartite graphs partitioning $E(M)$. □

Let G be a graph with $IB(G) = n$ and let $\{B_1, B_2, \ldots B_n\}$ be a set of induced bipartite subgraphs of G partitioning $E(G)$, with partitions $\{X_i, Y_i\}$. We define a homomorphism f of G to $EC(n)$ as follows. Given a vertex x of G, let $f(x)$ be a vertex (of $EC(n)$) whose ith coordinate is 0 if $x \in X_i$, is 1 if $x \in Y_i$, and is ∞ otherwise. It is straightforward to check that f is a homomorphism. Thus together with Lemma 4 we have:

Corollary 6 *The maximum possible chromatic number of a graph G with $IB(G) \leq n$ is $\chi(EC(n))$.*

Corollary 7 *The Alon-Saks-Seymour Conjecture for a given n is equivalent to the claim that every 1-ball subgraph of $EC(n)$ is $n+1$ colourable.*

Proof: Note that each 1-ball subgraph M of $EC(n)$ satisfies $CB(M) \leq n$ and thus, assuming the Alon-Saks-Seymour Conjecture, it has chromatic number at most $n+1$. On the other hand every graph G with $CB(G) \leq n$ has also $IB(G) \leq n$ and thus admits a homomorphism to $EC(n)$. The image of G under the homomorphism that we mentioned above is a 1-ball subgraph of $EC(n)$ which must have, by our assumption, chromatic number at most $n+1$. This induces a proper $n+1$-colouring on G as well. $\square$

This shows that the Alon-Saks-Seymour Conjecture is about the chromatic number of 1-ball subgraphs of $EC(n)$. As we will see, if we do not care about the isomorphisms that exist between 1-ball subgraphs, then there are exactly $2^{n(n-1)}$ subgraphs of $EC(n)$ needing to be checked. The solution to Problem 2 is just the chromatic number of $EC(n)$ itself. For these reasons we will study $EC(n)$ in subsequent sections of this article.

3 Subgraphs of extended cubes

3.1 Domination and 1-ball subgraphs

We introduce some notation first. For a vertex $x = (x_1, x_2, \ldots x_n)$ of $EC(n)$ we define the *rank* of x to be the number of non-infinity x_i, $i = 1, 2, \ldots n$. The 0-support of x, denoted by $S_0(x)$, is defined $S_0(x) = \{i \mid x_i = 0\}$. The 1-support, denoted by $S_1(x)$ is defined analogously. The *support* of a vertex x is $S(x) = S_0(x) \cup S_1(x)$, i.e., the set of non-infinity coordinates of x. The *rank* of a vertex is the cardinality of its support; a vertex of rank n, that is a vertex with no ∞ coordinate, is called a *full–rank* vertex. Given a pair of vertices x and y, we say that x is *dominated* by y if $S_0(x) \subseteq S_0(y)$ and $S_1(x) \subseteq S_1(y)$.

Lemma 8 *The graph $EC(n)$ is 2^n-colourable and has an independent set of* *size* $2^n - 1$.

Proof: Any 0-ball subgraph of $EC(n)$ is an independent set of $EC(n)$. A 0-ball subgraph obtained by the set of vertices dominated by a fixed full rank vertex is an independent set of size $2^n - 1$. For colouring note that c, defined by $c(x) = S_1(x)$, is a proper colouring of $EC(n)$. $\square$

We conjecture that the independence number of $EC(n)$ is $2^n - 1$ and that each independent set of size $2^n - 1$ is dominated by a full-rank vertex. This conjecture for $n \leq 4$ can be easily verified. For $n = 5$ it is verified using Maple's graph theory package. If our conjecture is true, then it will imply that $\chi(EC(n)) \geq (\frac{3}{2})^n$ because $\chi(G) \geq |V(G)|/\alpha(G)$ holds for every graph.

Given a vertex x of $EC(n)$, let $f_i(x)$ be a vertex obtained from x by switching 0 and 1 in the ith-coordinate, or keeping it the same if the ith coordinate is ∞. Note that f_i is an automorphism of $EC(n)$. Using these automorphisms, given a subgraph M of $EC(n)$ and a vertex x of M one can always find an isomorphic copy of M in which the image of x has an empty 1-support. With this notation we give another proof of the following lemma which was originally proved in [8].

Lemma 9 *Any 1-ball subgraph M of $EC(n)$ with a full rank vertex is $n + 1$ colourable.*

Proof: We may consider an isomorphic copy of M which contains the vertex $\vec{0} = (0, 0, \ldots 0)$. Since no vertex of M has Hamming distance 2 or more from $\mathbf{0}$ each vertex has 1-support of size at most 1. Therefore the colouring of $EC(n)$ given above, i.e., $c(x) = S_1(x)$, is a colouring which uses at most $n+1$ colours on M. $\square$

Observation 10 *Note that in a 1-ball subgraph M, if a vertex x is dominated by a vertex y, then every M-neighbour of x is also an M-neighbour of y.*

In finding the chromatic number of a 1-ball subgraph we may ignore the set of rank 1 vertices as the following lemma shows.

Lemma 11 *Assuming that the Alon-Saks-Seymour Conjecture is false, let n be the smallest integer for which the conjecture fails. Let M be a minimal 1-ball subgraph of $EC(n)$ with $\chi(M) \geq n + 2$. Then each vertex of M has rank at least 2.*

Proof: For a contradiction let x be a vertex in M that is of rank 1 and let x_i be the only nonzero coordinate of x. We may assume, without loss of generality, that $x_i = 1$. Since M is minimal, it has no vertex dominated by another vertex of M. Therefore, no other vertex of M has ith coordinate 1, otherwise x would be a dominated vertex. Now we may delete the ith coordinate from each vertex to obtain a subgraph M' of $EC(n-1)$. The minimality of n implies that M' is n-colourable, which induces an n-colouring of $M - x$, thus proving that M is $n+1$-colourable. □

3.2 Characterizing maximal 1-ball subgraphs

Let M be a maximal 1-ball subgraph of $EC(n)$, i.e., adding any other vertex results in a pair of vertices at Hamming distance greater than 1. By Corollary 7, to settle the Alon-Saks-Seymour Conjecture for a given n it will be sufficient to prove it for the set of maximal 1-ball subgraphs of $EC(n)$. In this subsection we will show that each maximal 1-ball subgraph is uniquely determined by its set of rank-2 vertices and is equivalent to a unique labeled tournament.

A rank-2 vertex x with non-infinity coordinates at i and j will be denoted by $(i, x_i)(j, x_j)$. There are total of $2n(n-1)$ rank 2 vertices in $EC(n)$. We will use D_2 to denote this set of rank 2 vertices, $D_2' = \{(i,0)(j,0),(i,1)(j,1)|1 \leq i < j \leq n\}$, and $D_2'' = \{(i,0)(j,1),(i,1)(j,0)|1 \leq i < j \leq n\}$. Each of D' and D'' has $n(n-1)$ elements. Given a maximal 1-ball subgraph M of $EC(n)$ and a pair of indices i and j, with $1 \leq i < j \leq n$, exactly one of the two rank-2 vertices $a = (i,1)(j,1)$ and $b = (i,0)(j,0)$ is in $V(M)$. To see that at least one of them must be in $V(M)$ note that if a is at Hamming distance 2 from a vertex a' in $V(M)$, then a' must agree with b on the ith and the jth coordinates. If b is also at Hamming distance 2 from a vertex b' in $V(M)$, then b' must agree with a on the ith and the jth coordinates. But then a' and b' are at Hamming distance at least 2, which contradicts the choice of M. Now since M is a maximal 1-ball subgraph of $EC(n)$, one of a or b must be in $V(M)$. Similarly, exactly one of $(i,1)(j,0))$ and $(i,0)(j,1)$ is in $V(M)$. Thus the vertex set of every maximal 1-ball contains exactly $n(n-1)$ rank-2 vertices. We will call the set of rank-2 vertices of M, the *base* of M. This set uniquely determines M.

Theorem 12 *Any maximal 1-ball subgraph is uniquely determined by its base.*

Proof: We have seen that any maximal 1-ball subgraph has a unique, well defined base, B_M, of cardinality $n(n-1)$ and that the vertices in the base are pairwise at Hamming distance at most 1. On the other hand, given any set B of $n(n-1)$ rank 2 vertices, pairwise at Hamming distance at most 1, we define a maximal 1-ball subgraph, M_B, which contains B. We define M_B to be set of all vertices in $EC(n)$ that are at Hamming distance at most 1 from all the vertices in B. This guarantees maximality.

We check that M_B forms a 1-ball subgraph. First note that for any pair of coordinates, i and j, the size of B forces us to have precisely one vertex from D' and one vertex from D'' with support $\{i, j\}$. To see that no pair of vertices of M_B are at Hamming distance 2 or more assume, for contradiction, that a and b are a pair of vertices in M_B with Hamming distance greater than 1. Thus there are coordinates i and j at which a and b are both not infinity and do not agree. Now $(i, a_i)(j, a_j)$ and $(i, b_i)(j, b_j)$ are a pair of rank-2 vertices at Hamming distance 2 and support $\{i, j\}$, and therefore exactly one of them, say $(i, a_i)(j, a_j)$, is in B. However this rank-2 vertex is also at Hamming distance 2 from b, contradicting the choice of elements in M_B.

Thus from a base, B, we construct a maximal 1-ball subgraph M_B which contains B. The containment and uniqueness of a base shows that two bases, B and B' cannot yield the same 1-ball subgraph. Two maximal 1-ball subgraphs, M and M', cannot share the same base, B because each would contain M_B which contradicts the maximality of M_B. □

Corollary 13 *There are exactly* $2^{n(n-1)}$ *maximal 1-ball subgraphs of* $EC(n)$*.*

Therefore, to prove the Alon-Saks-Seymour Conjecture for a given value of n it is enough to prove it for the $2^{n(n-1)}$ maximal 1-ball subgraphs of $EC(n)$. We should mention here that, first of all, there are many isomorphisms between distinct members of this set of subgraphs. Secondly, we do not need to check the chromatic number of the actual maximal 1-ball subgraph but rather, again by Observation 10, to check the chromatic number of the subgraph induced by the non-dominated set of vertices of the maximal 1-ball subgraph.

We now also show that each maximal 1-ball subgraph can be uniquely associated with a labeled tournament. Recall that the term *node* is used only for these directed tournaments and *vertex* refers to the undirected graphs $EC(n)$ and its subgraphs.

Definition 14 *A* 2-arc-labeled tournament *of order* n *is a directed graph with nodes* $\{1, 2, \ldots, n\}$ *and exactly one arc connecting nodes* i *and* j*. Each arc is labeled either* **0** *or* **1**.

We will often think of the labels as colours and refer to subtournaments, all of whose arcs have the same label, as *monochromatic*. We will denote labels of an arc in a tournament in a bolface, like **0** or **b**, and use these values without boldface when they are elements in a vector which is a vertex of $EC(n)$.

Let T be a 2-arc-labeled tournament of order n. For each arc (i, j) of T that has label **b** we associate a subset $S_{i,j,b} \subset V(EC(n))$ as

$$S_{i,j,b} = \{x \in V(EC(n)) \mid \text{ if } x_i = b \text{ then } x_j = \infty\}.$$

We now define the vertex set

$$V_T = \bigcap_{\substack{(i,j) \in A(T) \\ (i,j) \text{ labeled } \mathbf{b}}} S_{i,j,b}$$

and let M_T be the subgraph induced on this set of vertices.

Theorem 15 *For any 2-arc-labeled tournament of order n, T, M_T is a maximal 1-ball subgraph of $EC(n)$. For any maximal 1-ball subgraph of $EC(n)$, M, there is a unique 2-arc-labeled tournament of order n, T such that $M_T = M$.*

Proof: Suppose that $x, y \in V_T$ and they have Hamming distance at least 2. Thus there are coordinates i and j at which x and y are both not infinity and do not agree. Let the arc connecting i and j in T be (i,j) and be labeled $\mathbf{b}$. One of x or y contains a b in position i, say x. The fact that $x_j \neq \infty$ means $x \notin S_{i,j,b}$ and this contradicts the fact that $x \in V_T$. Thus M_1 is a 1-ball subgraph. We can determine the rank-2 vertices in V_T. For any arc $(i,j) \in A(T)$ labeled $\mathbf{b}$ it is clear that $(i, b+1)(j, 0)$ and $(i, b+1)(j, 1)$ are in V_T and $(i, b)(j, 0)$ and $(i, b)(j, 1)$ cannot be. If M_T is not maximal let $\overline{M_T}$ be a maximal 1-ball subgraph containing M_T. Then there is a vertex $z \in V(\overline{M_T}) \setminus V_T$. Since it is not in V_T, there must exist an arc $(i,j) \in A(T)$ labeled $\mathbf{b}$, with $z_i = b$ and $z_j \neq \infty$. The maximality of 1-ball subgraph $\overline{M_T}$ gives that exactly one of the two rank-2 vertices with support $\{i,j\}$ and value b in position i must be in $V(M_T)$. But from the corresponding arc in T we know that both rank-2 vertices with support $\{i,j\}$ and $b+1$ in position i are also in $V(M_T)$. But there is a pair at Hamming distance 2 among these three rank-2 vertices which contradicts the assumption that M_T was not maximal and this proves the first assertion.

To prove the second assertion, let M be a maximal 1-ball subgraph of $EC(n)$. We will define a 2-arc-labeled tournament, T_M, from M. We do this by using the rank-2 vertices of M to define both a direction and label on the arcs of the tournament. For each pair $i, j \in \{1, 2, \ldots, n\}$, there are exactly two rank-2 vertices in $V(M)$ with support $\{i,j\}$. These two are at Hamming distance 1 and so must have the same value, say b, in one coordinate, say i. In this case we include an arc directed (i,j) and labeled $\mathbf{b}$ in T_M. The rank-2 vertices of M and M_{T_M} are the same so therefor by Theorem 12 the uniqueness of the correspondence between maximal 1-ball subgraphs and 2-arc-labeled tournaments follows. □

Our correspondence between rank-2 vertices of M and the directions and labels of the arcs of T_M allows a nice description of some automorphisms of $EC(n)$ in terms of the tournaments. This description is given at the end of next section, after we determine all the automorphisms of $EC(n)$.

Another reformulation of the Alon-Saks-Seymour Conjecture comes from the following observation about independent sets in a 1-ball subgraph. While we do not have a characterization of all maximal independent sets of $EC(n)$, there is an easy characterization of independent sets in each 1-ball subgraph. Since there are no two vertices at Hamming distance 2 or more in a 1-ball subgraph and since every pair of vertices at Hamming distance exactly 1 are adjacent, the only independent sets in a 1-ball subgraph are 0-ball subgraphs. It is not hard to check that the set of vertices in any 0-ball subgraph is dominated by a full rank vertex. Thus we have the following reformulation of the Alon-Saks-Seymour Conjecture.

Proposition 16 *The Alon-Saks-Seymour Conjecture is equivalent to the statement that the vertices of every 1-ball subgraph of $EC(n)$ are dominated by a set of $n+1$ vertices in $EC(n)$.*

4 Automorphisms of $EC(n)$

Let a be a permutation of the n coordinates of the vertices of $EC(n)$; obviously a is an automorphism of $EC(n)$. Let f_i be a function which exchanges values of 0 and 1 in the ith coordinate of each vertex of $EC(n)$, leaving it the same if the ith coordinate is ∞. Again it is not hard to see that f_i is an automorphism of $EC(n)$. Composition of any number of these automorphisms is also an automorphism; let $\mathcal{A}(n)$ be the set of all these automorphisms. In this section we show that $\mathcal{A}(n)$ is the full automorphism group of $EC(n)$. We note that $\mathcal{A}(n)$ is isomorphic to $S_2(S_n)$, the wreath product of S_2 and S_n.

Let x be a vertex of rank i in $EC(n)$. Any vertex y adjacent to x can be formed from x by exchanging 0 and 1 in one entry (i choices), then replacing some subset of the other entries in $S(x)$ by ∞ (2^{i-1} choices), then setting each entry outside $S(x)$ arbitrarily (3^{n-i} choices). Hence, a vertex of rank i has degree $d_i = i2^{i-1}3^{n-i}$. We show that the degree uniquely corresponds to the rank.

Claim 17 *Suppose $i2^{i-1}3^{n-i} = j2^{j-1}3^{n-j}$ and $i > j$ are positive integers, then $i = 3$ and $j = 2$.*

Proof: We have $i/j = (3/2)^{i-j}$. Let $h = i - j \geq 1$, then

$$1 + \frac{h}{j} = \left(\frac{3}{2}\right)^h. \tag{1}$$

Since

$$\left(\frac{3}{2}\right)^h = e^{h\ln(3/2))} > 1 + h\ln(3/2) > 1 + 0.4h,$$

Equation (1) implies that $j \le 2$. The case $j = 1$ can be immediately excluded because, noting that $h \ge 1$, the right hand side of Equation (1) is a positive integer while the left hand side is not integral. So we only need to consider the case $j = 2$. Now the left hand side of Equation (1) is a half integer and hence the right hand side implies that $h = 1$. Indeed, $j = 2$ and $h = 1$ satisfy Equation (1) and this proves the claim. □

Since a vertex of rank i can be mapped to any other vertex of rank i using automorphisms from $\mathcal{A}(n)$, the set of vertices of rank i ($i \neq 2, 3$) forms an orbit under the action of the automorphism group of $EC(n)$ on vertices of $EC(n)$.

Let ϕ be any automorphism of $EC(n)$. Then ϕ must map a vertex of rank 1 to a vertex of rank 1. Let (i, x_i) be a vertex of rank 1 with $x_i \neq \infty$. Then the action of ϕ on $(i, 0)$ or $(i, 1)$ completely determines how ϕ acts on the ith coordinate of each vertex. If $\phi((i, 0))$ has a 1 in its non-zero coordinate, then ϕ must exchange every 0 and 1 in the ith coordinate of each vector in order to preserve adjacency. If the non-zero coordinate of $\phi((i, 0))$ is a 0, then ϕ should not make any change to the ith coordinate of any vector. Note that in either case if the ith coordinate of a vertex z is ∞, then ϕ should keep it as ∞, as otherwise $\phi(z)$ will be adjacent to either $\phi((i, 0))$ or $\phi((i, 1))$.

Similarly, if the lth coordinate of $\phi((i, 0))$ is the non-infinity coordinate, then ϕ must take the ith coordinate of each vertex to the lth coordinate (after the change from previous part). Moreover, this correspondence of i to l must be one-to-one as we are not allowed to create new adjacency. Hence we have proved:

Proposition 18 *The set $\mathcal{A}(n)$ of automorphisms, generated by permutations of coordinates and exchanges of 1 and 0 in a fixed coordinate, is the full automorphism group of $EC(n)$.*

Two bases are isomorphic if there is an automorphism of $EC(n)$ which maps one base to the other. We note that two bases are isomorphic if and only if their corresponding maximal 1-ball subgraphs are isomorphic. Isomorphism between two bases can also be described using their corresponding 2-arc-labeled tournaments, as follows: We note that a permutation of the coordinates of the vertices in $EC(n)$ simply corresponds to the same permutation of the nodes of the tournaments. The automorphism f_i, i.e., switching labels **0** and **1** at coordinate i, does not change the underlying unlabeled tournament, it simply switches the label of every arc leaving node i while keeping the labels of every other arc unchanged. This observation will be used in Section 5 to verify the Alon-Saks-Seymour Conjecture for small values of n.

5 The rank of 1-ball subgraphs

As Lemma 9 indicates, the existence of a vertex of large rank in a 1-ball subgraph leads to a small chromatic number. This leads to the following natural question: what is the minimum value of the maximum rank of a vertex over all maximal 1-ball subgraphs? For this purpose, we define the rank of a maximal 1-ball subgraph to be the maximum rank of its vertices, and let $\rho(n)$ denote the minimum rank of a maximal 1-ball subgraph. We now prove

Proposition 19 $\rho(n) = \Theta(\log n)$*, that is, there are positive constants* c_1 *and* c_2 *such that* $c_1 \log n \leq \rho(n) \leq c_2 \log n$*.*

Proof: This problem of finding $\rho(n)$ is closely related to the Ramsey number $R(m, m)$. Let $C(n)$ be the largest integer such that every 2-edge-coloured K_n contains a monochromatic clique of order $C(n)$ and there is a 2-edge-coloured K_n which does not contain a monochromatic clique of order $C(n)+1$. We note that

$$R(C(n), C(n)) \leq n < R(C(n) + 1, C(n) + 1).$$

Since $c_1 2^{m/2} \leq R(m, m) \leq 4^{m-1}$ for some positive constants c_1 (see, e.g., [9]), we have

$$c_1 2^{C(n)/2} \leq n \leq 4^{C(n)},$$

which implies

$$C(n) = \Theta(\log n).$$

Next we will show that $C(n) \leq \rho(n) \leq 2C(n)$ for every positive integer n. This immediately implies Proposition 19. When we refer to "monochromatic" we are thinking of the labels on the arcs as colours.

Let M be a maximal 1-ball subgraph of $EC(n)$ and T be the corresponding 2-arc-labeled tournament. A vertex x of M, with 1-support $S_1(x)$ and 0-support $S_0(x)$, corresponds to an ordered pair (T_0, T_1) of sub-tournaments of T, where T_0 is induced by $S_0(x)$ whose arcs all have label $\mathbf{0}$, T_1 is induced by $S_1(x)$ whose arcs all have label $\mathbf{1}$, all arcs in T from $S_0(x)$ to $S_1(x)$ have label $\mathbf{0}$ and all arcs in T from $S_1(x)$ to $S_0(x)$ have label $\mathbf{1}$. This observation will also be useful for generating all the vertices in M of rank greater than 2.

Now let M be a maximal 1-ball subgraph of $EC(n)$ whose rank is $\rho(n)$. Since the corresponding 2-arc-labeled tournament T contains a monochromatic sub-tournament of size $C(n)$, we have $\rho(n) \geq C(n)$.

On the other hand, let T be a 2-arc-labeled tournament of order n such that it does not contain a monochromatic sub-tournament of order $C(n)+1$. Let v be a vertex with maximum rank in the corresponding maximal 1-ball subgraph.

Then

$$\rho(n) \le |S_0(v)| + |S_1(v)| \le 2C(n).$$

This completes the proof of Proposition 19. □

6 Verifying the conjecture for small values of n

With the terminology from previous section we make further reductions, and using these reductions we can prove the conjecture for small values of n.

Let M be a maximal 1-ball subgraph of $EC(n)$ and $k < n$ be an integer. Let J be a k-subset of $\{1, 2, \ldots, n\}$. A set U of vertices in M is called J-independent if $|S_0(x) \cap S_1(y) \cap J| + |S_1(x) \cap S_0(y) \cap J| = 0$ for any two vertices x and y in U. We note that, if U is J-independent, then the subgraph $M[U]$ of M induced by U admits a homomorphism to a maximal 1-ball subgraph of $EC(n-k)$. Let $M[\bar{U}]$ be the subgraph of M induced by the complement of U, then we have

$$\chi(M) \le \chi(M[U]) + \chi(M[\bar{U}]).$$

Suppose that the Alon-Saks-Seymour Conjecture is true for all $k \le n-1$ and there is a subset U of $V(M)$ which is J-independent for some J, of size k, such that $\chi(M[\bar{U}]) \le k$. Then we have

$$\chi(M) \le k + (n - k + 1) = n + 1.$$

This observation is used to derive the following.

Proposition 20 *Suppose that the Alon-Saks-Seymour Conjecture is true for each $k < n$. Let T be a 2-arc-labeled tournament with n nodes such that one (or both) of the two monochromatic subgraphs of T contains a node with outdegree 0. Then the corresponding maximal 1-ball subgraph is $(n+1)$-colourable.*

Proof: We only need to consider the case when the monochromatic subgraph of T labeled **1** contains a node with outdegree 0, since the other case can be converted to the first case by switching 0 and 1 at every coordinate. Let T_1 be the monochromatic subgraph of T with label **1** and suppose that node i has outdegree 0 in T_1. Let M be the maximal 1-ball subgraph corresponding to T, and U be the subset of $V(M)$ consisting of all the vertices whose ith coordinate is either 0 or ∞. Then $\bar{U}$, the complement of U, must form an independent set in M. If $\bar{U}$ contains two adjacent vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, then there is a coordinate $j \ne i$ such that $\{x_j, y_j\} = \{0, 1\}$. This implies that T_1 contains the arc directed from i to j, contradicting the assumption that node i has outdegree 0 in T_1. By the above observation, $\chi(M) \le 1 + n$. □

Thus we only need to consider 2-arc-labeled tournaments T such that at every node there is an outbound arc of label $\mathbf{0}$ and an outbound arc of label $\mathbf{1}$. In particular, we only need to consider those tournaments in which every node has outdegree at least 2. This observation already proves the conjecture for $n \leq 5$.

Corollary 21 *The Alon-Saks-Seymour Conjecture is true for $n \leq 5$.*

Proof: We note that $EC(1)$ is isomorphic to K_2, and hence the conjecture is true for $n = 1$. For $n \leq 4$, every tournament contains a node whose outdegree is less than 2 (otherwise it needs at least $2n$ arcs), thus we may apply Proposition 20 and induction. We also note that the case $n \leq 4$ was proved by Rho [8].

Next we consider the case $n = 5$. In this case either Proposition 20 will apply again or each node has 2 outbound arcs, one with each label. We claim that each such 2-arc-labeled tournament is the basis of a 1-ball subgraph isomorphic to the one obtained from two monochromatic 5-cycles. To see this, suppose T is a 2-arc-labeled tournament on 5 nodes with each node having exactly 2 outbound arcs one with each label. If each node also has an incoming arc with each label then T must be union of two directed monochromatic 5-cycles and we are done.

Otherwise there is a node v_1 with two incoming arcs labeled $\mathbf{0}$. Let v_2v_1, v_3v_1, v_1v_4 be the arcs labeled $\mathbf{0}$ incident to v_1, thus v_1v_5 is labeled $\mathbf{1}$. By the automorphism f_1 we have symmetry between v_4 and v_5 so we may assume v_5v_4 is the arc connecting v_4 and v_5. Therefore of the two arcs connecting v_2 and v_3 to v_5 one must be an incoming arc of v_5 (say v_2v_5) and it must be labeled $\mathbf{1}$. Now applying f_2 will balance the labels of the incoming arcs at v_1 and v_5 and not change the number or labels of incoming nor outgoing arcs at any other node. By repeating this if needed we conclude that each vertex also has one incoming arc of each label. Now the 1-ball subgraph obtained from this will have the following 10 vertices as the only non-dominated vertices.

Rank 2 vertices:
$(0, 0, \infty, \infty, \infty), (\infty, 0, 0, \infty, \infty), (\infty, \infty, 0, 0, \infty), (\infty, \infty, \infty, 0, 0), (0, \infty, \infty, \infty, 0)$.
Rank 3 vertices:
$(0, 1, \infty, 1, \infty), (\infty, 0, 1, \infty, 1), (1, \infty, 0, 1, \infty), (\infty, 1, \infty, 0, 1), (1, \infty, 1, \infty, 0)$.

A 6-colouring is obtained easily by colouring all the rank 2 vertices with one colour assigning each rank 3 vertex a new distinct colour. Note that rank-2 vertices form an independent set as they are dominated by $(0, 0, 0, 0, 0)$. □

With the aid of a long computation we were able to verify this conjecture for $n \leq 9$. The use of Proposition 20 to prove the case for n depends on the truth of the conjecture for all $k < n$. In the reminder of this section we describe the computational methods for $n = 9$, this computation depends on $n = 6, 7, 8$ which were checked with the same methods.

Note that for $n = 9$, the number of 1-ball subgraphs needed to be checked is 2^{72} and, with today's computational power, it is practically impossible to even produce all these graphs. However there are many isomorphisms among maximal 1-ball subgraphs and the task is tractable once most isomorphic copies are removed. To this end we first start with a list of the 191536 non-isomorphic tournaments on 9 nodes.

Now, given a fixed tournament, T, on 9 nodes, we have 2^{36} possible 2-arc-labelings, each corresponding to a maximal 1-ball subgraph. However there are still many isomorphisms among these maximal 1-ball subgraphs. Such an isomorphism may be induced by an automorphism f_i of $EC(n)$ for $i = 1, 2, \ldots 9$, by an automorphism of T itself, or by any combination of these. We note that the actions of f_i on a maximal 1-ball subgraph is equivalent to exchanging the label of the outbound arcs from node i in the corresponding 2-arc-labeled tournament. If we pick an outbound arc for each node in T and then fix a label for that arc, we still will produce all the non-isomorphic maximal 1-ball subgraphs obtained form T, but with far fewer isomorphic copies. Those labelings for which all the arcs leaving some node have the same label can be discarded, using Proposition 20.

Our next step was to produce and then colour the 1-ball subgraph from a given 2-arc-labeled tournament. To make the graphs smaller, and so easier to colour, we discarded the dominated vertices as we discussed earlier. To produce the set of all non-dominated vertices from a given 2-arc-labeled tournament we introduced the concept of bident-free subtournament. A *bident* in a 2-arc-labeled tournament is subgraph consisting of 2 arcs with different labels and common beginning point. A *subtournament* of a given tournament is an induced sub-digraph on any subset of nodes of the tournament. A bident-free subtournament in a 2-arc-labeled tournament is a subtournament that has no bident.

If a maximal 1-ball subgraph M is encoded by a 2-arc-labeled tournament T_M, then each vertex v of M will correspond to a subtournament T_v which is the union of a clique T_0 labeled $\mathbf{0}$ (nodes corresponding to 0 coordinates) and a clique T_1 labeled $\mathbf{1}$ (nodes corresponding to 1 coordinates) with the extra property that each outbound arc from T_0 is labeled $\mathbf{0}$ and each outbound arc from T_1 is labeled $\mathbf{1}$. It is now clear that T_v is bident-free. On the other hand given a bident-free subtournament T' we can construct one or two vertices corresponding to T'. To this end, for each coordinate, x, if x is not a node of T' choose ∞ for x, if x is a node of T' with an outbound arc with label $\mathbf{0}$ then choose 0 for this coordinate, if x has an outbound arc with label $\mathbf{1}$ choose 1 for this coordinate, if x has no outbound arc then x can be either 0 or 1, in which case we will have two vertices of M corresponding to T'.

Note that of the two vertices obtained in the last case, it is possible to have a non-dominated vertex and a dominated vertex. Thus it is not necessarily true that the bident-free tournament corresponding to a non-dominated vertex is maximal with respect to being bident-free, but this is almost true. We use the set of maximal bident-free subtournaments to produce our set of non-dominated vertices as follows.

We first find all the maximal bident-free subtournaments. This was achieved using the following iteration. For disjoint $C, L, F \subseteq \{1, 2, \ldots, n\}$, let $\langle C, L, F\rangle$ denote the set of all bident-free sets X such that $C \subseteq X \subseteq C \cup L$ and such that $X + x$ is not bident-free for any $x \in (L \setminus X) \cup F$. The set of all maximal bident-free subsets is $\langle \emptyset, \{1, 2, \ldots, 9\}, \emptyset\rangle$. We determine the elements of this set by repeatedly applying the refinement $\langle C, L, F\rangle = \langle C+x, L-x, F\rangle \cup \langle C, L-x, F+x\rangle$ for some $x \in L$. At any step we can apply the following simplification rules:

(1) $\langle C, L, F\rangle = \langle C, L-x, F\rangle$ for any $x \in L$ such that $C+x$ is not bident-free.
(2) $\langle C, L, F\rangle = \langle C, L, F-x\rangle$ for any $x \in F$ such that $C+x$ is not bident-free.
(3) $\langle C, L, F\rangle = \emptyset$ if $C \cup L+x$ is bident-free for any $x \in F$.
(4) $\langle C, L, F\rangle = \{C \cup L\}$ if rules 1–3 do not apply and $C \cup L$ is bident-free.

The set of non-dominated vertices are now built as follows. First for each maximal bident-free subtournament we construct one or two vertices corresponding to it. Then for each node x of a maximal bident-free subtournament T', we remove all the outbound arcs together with their end nodes to obtain a bident-free subtournament T'_x in which x is a sink. Then we add to L the two vertices made from T'_x if they are not dominated. Finally, we remove duplicate vertices by sorting; usually there were very few.

For each graph M produced by this method, we applied the standard greedy colouring algorithm with random vertex labelings until a colouring with 10 colours was found. This was successful in all cases.

Of 191536 nonisomorphic tournaments on 9 nodes, 6880 have minimum outdegree 0 and 50816 have minimum outdegree 1. There are 105916 tournaments of minimum outdegree 2, 27909 tournaments of minimum outdegree 3, and 15 of minimum outdegree 4.

Our computation produced 2709401599952 graphs with 21 being the number of vertices of the smallest graph and 66 being the number of vertices of the largest graph. Table 6 shows how many graphs there were of each order. The total time for the computation was 11 GHz–years.

$\|V(M)\|$	Number of graphs	$\|V(M)\|$	Number of graphs
21	1	44	125901710465
22	604	45	81267251957
23	22546	46	48642558565
24	335917	47	27104914998
25	3009042	48	14110182128
26	19453912	49	6883192715
27	99243002	50	3150968376
28	414422072	51	1356263082
29	1449107249	52	548904802
30	4319529292	53	209052332
31	11117985007	54	74877323
32	24970401600	55	25204691
33	49387718809	56	7956173
34	86792501140	57	2363278
35	136716885660	58	652342
36	194566281907	59	167827
37	251704504252	60	40896
38	297176474277	61	8294
39	320821496578	62	1722
40	316875725672	63	207
41	286424162481	64	34
42	237128311702	65	1
43	180127755021	66	1
Total	2709401599952		

Table 1. Colouring 1-ball subgraphs of $EC(9)$

7 Concluding Remarks

This paper makes several contributions to the investigation of the Alon-Saks-Seymour Conjecture:

- With our definitions and restatements of the problem in terms of $EC(n)$, all previously solved cases of the conjecture and the case $n = 5$ can be succinctly and combinatorially proven without case analysis;
- We explicitly show that the number of cases to check for any n is finite;
- We have expressed the problem in terms of graph homomorphisms which is interesting in and of itself.

We note that N. Alon and I. Haviv [1] and, independently, D. Mubayi and S. Vishwanathan [7] have proved that for any graph G with $CB(G) \leq n$ we have $\chi(G) \leq n^{\frac{1}{2}\log_2 n - 1}$. A similar result is obtained by K. Fischer and Z. Dvorak by improving Proposition 20 and by improving the assignment of $x \to S_0(x)$.

It is natural to think that the more non-dominated vertices we have in a maximal 1-ball subgraph the more chances we have to find a counterexample to the Alon-Saks-Seymour Conjecture, if a counterexample exists. In this regard we would like to ask what is the largest size of a 1-ball subgraph of $EC(n)$ with no dominated vertices. For $n = 5$ and $n = 9$ we have observed that such a largest graph is obtained when the 2-arc-labeling in the corresponding tournament comes from a quadratic residue graph, i.e., $\{x, y\}$ is labeled $\mathbf{0}$ if $x - y$ is a square in $\mathbb{Z}_n$ and labeled $\mathbf{1}$ otherwise. We note that for these orders this is also a Ramsey colouring, i.e., a colouring in which the largest monochromatic subgraph is as small as possible. The arcs in these tournaments are directed from x to $x + i \pmod{n}$ for $i < n/2$. We do not know if this is a general pattern. But we propose that perhaps answering the following question will settle the Alon-Saks-Seymour Conjecture in the negative:

Problem 22 *Given $n = 4p + 1$ where p is a prime power, what is the largest chromatic number of a 1-ball subgraph of $EC(n)$ corresponding to a 2-arc-labeled Eulerian tournament whose labels correspond to the the quadratic residue graph?*

We note that the origin of the Graham-Pollak theorem comes from studying the possibility of embedding a graph G in $EC(n)$ for some n. By proving $\omega(EC(n)) = n + 1$ they showed that the smallest n might be as big as $V(G)-1$. M. Winkler, proving the Graham-Pollak conjecture [10], showed that $n = V(G) - 1$ works for every graph. Perhaps studying $EC(n)$ would help to find a combinatorial proof of the Graham-Pollak theorem. So we propose the following question:

Problem 23 *What is $\alpha(EC(n))$?.*

We believe that it is $2^n - 1$.

Finally, we strongly believe that this combinatorial casting of the problem using $EC(n)$ and its subgraphs can lead to new advances towards the conjecture in general and is a likely foundation for further research because it gives a powerful framework which can motivate and translate new techniques towards solving this problem. In particular we note that Alon and Haviv's bound on the chromatic number uses this framework and similarly with Fischer and Dvorak's advances. We think it is likely that Proposition 20 can be pushed further, likely by further considering the structures of bident-free tournaments.

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