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Infinite products with coefficients which vanish on certain arithmetic progressions

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Let q denote a complex variable with $|q| < 1$. For a positive integer k let

$$E_k = E_k(q) := \prod_{n=1}^{\infty}(1 - q^{kn}).$$

If $f(q) = \sum_{n=0}^{\infty} f_n q^n$ we define $[f(q)]_n := f_n$ for each nonnegative integer n. In this paper, we determine results of the type

$$[E_1^{-8}E_2^{24}]_{4k+3} = 0, \quad k = 0, 1, 2, 3, \ldots.$$

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1. Introduction

Let $\mathbb{Z}$ denote the set of all integers, $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0$ the set of nonnegative integers. Throughout this paper q denotes a complex variable with $|q| < 1$. For $k \in \mathbb{N}$ we define

$$E_k = E_k(q) := \prod_{n\in\mathbb{N}}(1 - q^{kn}). \tag{1.1}$$

We note that $E_1(q^k) = E_k(q)$. Using MAPLE we find that

$$E_1^{-8}E_2^{24} = 1 + 8q + 20q^2 - 78q^4 - 128q^5 - 104q^6 + 455q^8 + 832q^9 + 260q^{10} - 290q^{12} - 2560q^{13} - 1864q^{14} - 3393q^{16} + 2080q^{17} + O(q^{18}).$$

This expansion suggests that the coefficients of q^{4k+3} ($k \in \mathbb{N}_0$) in the power series expansion in powers of q of the infinite product $E_1^{-8}E_2^{24}$ are all zero, that is

$$[E_1^{-8}E_2^{24}]_{4k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0, \tag{1.2}$$

where $[f(q)]_n$ denotes the coefficient f_n in $f(q) = \sum_{n=0}^{\infty} f_n q^n$.

In this paper, we consider the problem of determining for each $t \in \{1, 2, 3, 4\}$ all the integers a and b such that

$$[E_1^a E_2^b]_{4k+t} = 0 \quad \text{for all } k \in \mathbb{N}_0,$$

and for each $t \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ all the integers a, b and c such that

$$[E_1^a E_2^b E_4^c]_{8k+t} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

In Sec. 3 and 4, we prove the following two theorems.

Theorem 1.1. *Let $a, b \in \mathbb{Z}$. Then*

(i) $[E_1^a E_2^b]_{4k+1} = 0$ *for all* $k \in \mathbb{N}_0$ *if and only if*

$$(a, b) = (0, b);$$

(ii) $[E_1^a E_2^b]_{4k+2} = 0$ *for all* $k \in \mathbb{N}_0$ *if and only if*

$$(a, b) = (-4, 14),\ (-2, 5),\ (0, 0),\ (2, -1),\ (4, 2);$$

(iii) $[E_1^a E_2^b]_{4k+3} = 0$ *for all* $k \in \mathbb{N}_0$ *if and only if*

$$(a, b) = (-8, 24),\ (-4, 10),\ (-2, 5),\ (0, b),\ (2, -1),\ (4, -2),\ (8, 0);$$

(iv) $[E_1^a E_2^b]_{4k+4} = 0$ *for all* $k \in \mathbb{N}_0$ *if and only if*

$$(a, b) = (0, 0).$$

Theorem 1.2. *Let $a, b, c \in \mathbb{Z}$. Then*

(i) $[E_1^a E_2^b E_4^c]_{8k+1} = 0$ *for all* $k \in \mathbb{N}_0$ *if and only if*

$$(a, b, c) = (0, b, c);$$

(ii) $[E_1^a E_2^b E_4^c]_{8k+2} = 0$ *for all* $k \in \mathbb{N}_0$ *if and only if*

$$(a, b, c) = (-6, 27, -6),\ (-4, 14, c),\ (-2, 5, c),\ (0, 0, c),$$
$$(2, -1, c),\ (4, 2, c),\ (6, 9, 0);$$

(iii) $[E_1^a E_2^b E_4^c]_{8k+3} = 0$ *for all* $k \in \mathbb{N}_0$ *if and only if*

$$(a, b, c) = (-10, 33, -10),\ (-8, 24, c),\ (-4, 10, c),\ (-2, 5, c),$$
$$(0, b, c),\ (2, -1, c),\ (4, -2, c),\ (8, 0, c),\ (10, 3, 0);$$

(iv) $[E_1^aE_2^bE_4^c]_{8k+4}=0$ *for all* $k\in\mathbb{N}_0$ *if and only if*

$$(a,b,c)=(-6,15,0),\ (-4,10,0),\ (-4,12,-9),\ (-4,14,-14),\ (-4,16,-15),\ (-4,18,-12),\ (-2,1,14),\ (-2,3,5),\ (-2,5,0),\ (-2,7,-1),\ (-2,9,2),\ (0,-4,14),\ (0,-2,5),\ (0,0,0),\ (0,2,-1),\ (0,4,2),\ (2,-5,16),\ (2,-3,7),\ (2,-1,2),\ (2,1,1),\ (2,3,4),\ (4,-2,4),\ (4,0,-5),\ (4,2,-10),\ (4,4,-11),\ (4,6,-8),\ (6,-3,6);$$

(v) $[E_1^aE_2^bE_4^c]_{8k+5}=0$ *for all* $k\in\mathbb{N}_0$ *if and only if*

$$(a,b,c)=(-8,20,-2),\ (-8,22,-11),\ (-8,24,-16),\ (-8,26,-17),\ (-8,28,-14),\ (-6,15,-2),\ (-4,6,12),\ (-4,8,3),\ (-4,10,-2),\ (-4,12,-3),\ (-4,14,0),\ (-2,1,12),\ (-2,3,3),\ (-2,5,-2),\ (-2,7,-3),\ (-2,9,0),\ (0,b,c),\ (2,-5,14),\ (2,-3,5),\ (2,-1,0),\ (2,1,-1),\ (2,3,2),\ (4,-6,16),\ (4,-4,7),\ (4,-2,2),\ (4,0,1),\ (4,2,4),\ (6,-3,4),\ (8,-4,6),\ (8,-2,-3),\ (8,0,-8),\ (8,-2,-9),\ (8,4,-6);$$

(vi) $[E_1^aE_2^bE_4^c]_{8k+6}=0$ *for all* $k\in\mathbb{N}_0$ *for the following values of* (a,b,c)

$$(a,b,c)=(-12,34,-18),\ (-10,25,-4),\ (-10,33,0),\ (-8,20,-4),\ (-6,11,10),\ (-6,15,-4),\ (-6,19,-2),\ (-4,6,10),\ (-4,10,-4),\ (-4,12,-9),\ (-4,14,c),\ (-4,16,-15),\ (-4,18,-16),\ (-4,22,-14),\ (-2,-3,24),\ (-2,1,10),\ (-2,3,5),\ (-2,5,c),\ (-2,7,-1),\ (-2,9,-2),\ (-2,13,0),\ (0,-8,24),\ (0,-4,10),\ (0,-2,5),\ (0,0,c),\ (0,2,-1),\ (0,4,-2),\ (0,8,0),\ (2,-9,26),\ (2,-5,12),\ (2,-3,7),\ (2,-1,c),\ (2,1,1),\ (2,3,0),\ (2,7,2),\ (4,-6,14),\ (4,-2,0),\ (4,0,-5),\ (4,2,c),\ (4,4,-11),\ (4,6,-12),\ (4,10,-10),\ (6,-7,16),\ (6,-3,2),\ (6,1,4),\ (8,-4,4),\ (10,-5,6),\ (10,3,10),\ (12,-2,-6);$$

(vii) $[E_1^aE_2^bE_4^c]_{8k+7}=0$ *for all* $k\in\mathbb{N}_0$ *for the following values of* (a,b,c)

$$(a,b,c)=(-16,44,-20),\ (-12,30,-6),\ (-10,25,-6),\ (-8,16,8),$$

$$(-8,20,-6),\ (-8,22,-11),\ (-8,24,c),\ (-8,26,-17),$$
$$(-8,28,-18),\ (-8,32,-16),\ (-6,11,8),\ (-6,15,-6),$$
$$(-6,19,-4),\ (-6,27,0),\ (-4,2,22),\ (-4,6,8),\ (-4,8,3),$$
$$(-4,10,c),\ (-4,12,-3),\ (-4,14,-4),\ (-4,18,-2),$$
$$(-2,-3,22),\ (-2,1,8),\ (-2,3,3),\ (-2,5,c),\ (-2,7,-3),$$
$$(-2,9,-4),\ (-2,13,-2),\ (0,b,c),\ (2,-9,24),\ (2,-5,10),$$
$$(2,-3,5),\ (2,-1,c),\ (2,1,-1),\ (2,3,-2),\ (2,7,0),\ (4,-10,26),$$
$$(4,-6,12),\ (4,-4,7),\ (4,-2,c),\ (4,0,1),\ (4,2,0),\ (4,6,2),$$
$$(6,-7,14),\ (6,-3,0),\ (6,1,2),\ (6,9,6),\ (8,-8,16),\ (8,-4,2),$$
$$(8,-2,-3),\ (8,0,c),\ (8,2,-9),\ (8,4,-10),\ (8,8,-8),\ (10,-5,4),$$
$$(12,-6,6),\ (16,-4,-4);$$

(viii) $[E_1^a E_2^b E_4^c]_{8k+8} = 0$ *for all* $k \in \mathbb{N}_0$ *if*

$$(a,b,c) = (0,0,0).$$

We remark that (1.2) is the case $(a,b) = (-8,24)$ of Theorem 1.1(iii).

We note that all parts of Theorems 1.1 and 1.2 are "if and only if" statements except parts (vi), (vii) and (viii) of Theorem 1.2. This is because for these three parts we cannot be sure that MAPLE finds all the solutions in integers of certain sets of equations.

Preliminary results are proved in Sec. 2. Theorem 1.1 is proved in Sec. 3 and Theorem 1.2 in Sec. 4. The proofs of Theorems 1.1 and 1.2 are carried out in the same uniform manner and make use of basic identities satisfied by Ramanujan's theta function $\varphi(q)$, which is defined in Sec. 2.

2. Preliminary Results

Let

$$f(q) = \sum_{n=0}^{\infty} f_n q^n,$$

and

$$g(q) = \sum_{n=0}^{\infty} g_n q^n.$$

Let $s \in \mathbb{N}$ satisfy $s \geq 2$. Let $t \in \{1, 2, \ldots, s-1\}$. Our first result (Lemma 2.1) shows that if

$$[f(q)]_{sk+t} = 0 \quad \text{for all } k \in \mathbb{N}_0$$

then

$$[f(q)g(q^s)]_{sk+t} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

This simple result proves to be useful in the following way. For example we show in Sec. 4 that

$$[E_1^{-6}E_2^{27}E_4^{-6}E_8E_{16}^{-2}]_{8k+2} = 0 \quad \text{for all } k \in \mathbb{N}_0,$$

so that taking $s = 8$, $t = 2$, $f(q) = E_1^{-6}E_2^{27}E_4^{-6}E_8E_{16}^{-2}$ and $g(q) = E_1^{-1}E_2^2$, in Lemma 2.1, we obtain for all $k \in \mathbb{N}_0$

$$[E_1^{-6}E_2^{27}E_4^{-6}]_{8k+2} = [E_1^{-6}E_2^{27}E_4^{-6}E_8E_{16}^{-2} \times E_8^{-1}E_{16}^2]_{8k+2} = [f(q)g(q^8)]_{8k+2} = 0,$$

as $[f(q)]_{8k+2} = 0$. In other words Lemma 2.1 enables us to remove $E_8E_{16}^{-2}$ from the identity $[E_1^{-6}E_2^{27}E_4^{-6}E_8E_{16}^{-2}]_{8k+2} = 0$, which is valid for all $k \in \mathbb{N}_0$, to obtain $[E_1^{-6}E_2^{27}E_4^{-6}]_{8k+2} = 0$ for all $k \in \mathbb{N}_0$.

Lemma 2.1. *Let $s \in \mathbb{N}$ satisfy $s \geq 2$. Let $t \in \{1, 2, \ldots, s-1\}$. Suppose that*

$$f(q) = \sum_{l=0}^{\infty} f_l q^l$$

is such that

$$[f(q)]_{sk+t} = 0 \quad \textit{for all } k \in \mathbb{N}_0.$$

Let

$$g(q) = \sum_{m=0}^{\infty} g_m q^m.$$

Then

$$[f(q)g(q^s)]_{sk+t} = 0 \quad \textit{for all } k \in \mathbb{N}_0.$$

Proof. For $q \in \mathbb{C}$ with $|q| < 1$ we have

$$g(q^s) = \sum_{m=0}^{\infty} g_m q^{sm}.$$

Hence

$$f(q)g(q^s) = \sum_{n=0}^{\infty} \left(\sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+sm=n}} f_l g_m \right) q^n.$$

Thus for all $k \in \mathbb{N}_0$ we have

$$[f(q)g(q^s)]_{sk+t} = \sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+sm=sk+t}} f_l g_m.$$

As $l \geq 0$ and $t \leq s-1$ we have

$$sm \leq l + sm = sk + t \leq sk + s - 1 < sk + s$$

so that

$$m < k+1$$

and thus

$$0 \le m \le k.$$

Define a new integral variable h by $h = k - m$ so that

$$l = sh + t, \quad m = k - h,$$

with

$$0 \le h \le k.$$

Hence

$$[f(q)g(q^s)]_{sk+t} = \sum_{h=0}^{k} f_{sh+t} g_{k-h}.$$

But $f_{sh+t} = 0$ for all $h \in \mathbb{N}_0$, so

$$[f(q)g(q^s)]_{sk+t} = 0 \quad \text{for all } k \in \mathbb{N}_0,$$

as asserted. □

Our next result gives the well-known criterion on $f(q)$ which ensures that $[f(q)]_{sk+t} = 0$ for all $k \in \mathbb{N}_0$.

Lemma 2.2. *Let $s \in \mathbb{N}$ satisfy $s \ge 2$. Let $t \in \{1, 2, \ldots, s-1\}$. Let $\omega = e^{2\pi i/s}$. Suppose that*

$$f(q) = \sum_{n=0}^{\infty} f_n q^n$$

satisfies

$$\sum_{r=0}^{s-1} \omega^{-tr} f(\omega^r q) = 0.$$

Then

$$[f(q)]_{sk+t} = 0 \quad \textit{for all } k \in \mathbb{N}_0.$$

Proof. For $r \in \{0, 1, 2, \ldots, s-1\}$ we have

$$f(\omega^r q) = \sum_{n=0}^{\infty} f_n \omega^{rn} q^n.$$

Thus

$$\sum_{r=0}^{s-1} \omega^{-tr} f(\omega^r q) = \sum_{r=0}^{s-1} \omega^{-tr} \sum_{n=0}^{\infty} f_n \omega^{rn} q^n = \sum_{n=0}^{\infty} f_n q^n \sum_{r=0}^{s-1} \omega^{(n-t)r} = s \sum_{\substack{n=0 \\ n \equiv t \ (\mathrm{mod}\ s)}}^{\infty} f_n q^n.$$

Change the summation variable from $n \in \mathbb{N}_0$ satisfying $n \equiv t \pmod{s}$ to $k = (n-t)/s \in \mathbb{Z}$. As $n \geq 0$ and $t \leq s-1$ we have $k \geq \frac{-t}{s} \geq \frac{-(s-1)}{s} > -1$ so that $k \in \mathbb{N}_0$. Hence

$$\sum_{r=0}^{s-1} \omega^{-tr} f(\omega^r q) = s \sum_{k=0}^{\infty} f_{sk+t} q^{sk+t}.$$

Thus

$$\sum_{r=0}^{s-1} \omega^{-tr} f(\omega^r q) = 0 \quad \text{for all } q \in \mathbb{C},\ |q| < 1,$$

implies

$$[f(q)]_{sk+t} = 0 \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

We use Lemma 2.2 with $s = 4$ and $s = 8$. For $t \in \{1, 2, 3\}$ we have

$$\sum_{r=0}^{3} i^{3tr} f(i^r q) = 0 \Rightarrow [f(q)]_{4k+t} = 0 \quad \text{for all } k \in \mathbb{N}_0, \tag{2.1}$$

and for $t \in \{1, 2, 3, 4, 5, 6, 7\}$ we have

$$\sum_{r=0}^{7} \omega^{7tr} f(\omega^r q) = 0 \Rightarrow [f(q)]_{8k+t} = 0 \quad \text{for all } k \in \mathbb{N}_0, \tag{2.2}$$

where $\omega = e^{2\pi i/8} = \frac{1+i}{\sqrt{2}}$.

We make use of Ramanujan's theta functions

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

and

$$\psi(q) := \sum_{n=-\infty}^{\infty} q^{n(n+1)/2},$$

which are defined for $q \in \mathbb{C}$ with $|q| < 1$. The infinite product representations of $\varphi(q)$ and $\psi(q)$ are due to Jacobi, namely,

$$\varphi(q) = \frac{E_2^5}{E_1^2 E_4^2}, \quad \varphi(-q) = \frac{E_1^2}{E_2}, \quad \psi(q) = \frac{E_2^2}{E_1}. \tag{2.3}$$

We use the following four properties of $\varphi(q)$ and $\psi(q)$, all of which can be found in Berndt's book [1], namely

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{2.4}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8), \tag{2.5}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{2.6}$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2). \tag{2.7}$$

We use the relationships (2.4)–(2.7) to parametrize $\varphi(q^2)$, $\varphi(-q^2)$, $\varphi(q^4), \ldots$ in terms of $A := \varphi(q)$ and $B := \varphi(-q)$.

Lemma 2.3.

$$\begin{aligned}
\varphi(q) &= A,\\
\varphi(-q) &= B,\\
\varphi(q^2) &= \left(\frac{A^2+B^2}{2}\right)^{1/2},\\
\varphi(-q^2) &= (AB)^{1/2},\\
\varphi(q^4) &= \frac{A+B}{2},\\
\varphi(-q^4) &= \left(\frac{AB(A^2+B^2)}{2}\right)^{1/4},\\
\varphi(iq) &= \left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right) i,\\
\varphi(-iq) &= \left(\frac{A+B}{2}\right) - \left(\frac{A-B}{2}\right) i,\\
\varphi(iq^2) &= \left(\frac{\left(\frac{A^2+B^2}{2}\right)^{1/2} + (AB)^{1/2}}{2}\right) + \left(\frac{\left(\frac{A^2+B^2}{2}\right)^{1/2} - (AB)^{1/2}}{2}\right) i,\\
\varphi(-iq^2) &= \left(\frac{\left(\frac{A^2+B^2}{2}\right)^{1/2} + (AB)^{1/2}}{2}\right) - \left(\frac{\left(\frac{A^2+B^2}{2}\right)^{1/2} - (AB)^{1/2}}{2}\right) i.
\end{aligned}$$

Let $\omega := e^{2\pi i/8}$ *so that* $\omega^2 = i$, $\omega^3 = i\omega$, $\omega^4 = -1$, $\omega^5 = -\omega$, $\omega^6 = -i$, $\omega^7 = -i\omega$, $\omega^8 = 1$. *Then*

$$\begin{aligned}
\varphi(\omega q) &= \left(\frac{AB(A^2+B^2)}{2}\right)^{1/4} + \left(\frac{A-B}{2}\right)\omega,\\
\varphi(i\omega q) &= \left(\frac{AB(A^2+B^2)}{2}\right)^{1/4} + \left(\frac{A-B}{2}\right)\omega i,\\
\varphi(-\omega q) &= \left(\frac{AB(A^2+B^2)}{2}\right)^{1/4} - \left(\frac{A-B}{2}\right)\omega,\\
\varphi(-i\omega q) &= \left(\frac{AB(A^2+B^2)}{2}\right)^{1/4} - \left(\frac{A-B}{2}\right)\omega i.
\end{aligned}$$

Proof. The formulas for $\varphi(q^2)$, $\varphi(-q^2)$ and $\varphi(q^4)$ follow from (2.7), (2.6) and (2.4), respectively. The formula for $\varphi(-q^4)$ follows from (2.6) with q replaced by q^2 and the formulas for $\varphi(q^2)$ and $\varphi(-q^2)$. The formulas for $\varphi(iq)$ and $\varphi(-iq)$ follow from (2.4) and (2.5) with q replaced by iq. The formulas for $\varphi(iq^2)$ and $\varphi(-iq^2)$ follow from those for $\varphi(iq)$ and $\varphi(-iq)$ on replacing q by q^2 and using

$$A(q^2) = \varphi(q^2) = \left(\frac{A^2+B^2}{2}\right)^{1/2}, \quad B(q^2) = \varphi(-q^2) = (AB)^{1/2}.$$

The final four formulas follow from (2.4) and (2.5) with q replaced by ωq and $i\omega q$. □

Our next result uses Lemmas 2.1–2.3 to show that the verification of a certain identity in two complex variables z and w for $r, s \in \mathbb{N}_0$ and $t \in \{1, 2, 3, 4\}$ establishes that $[E_1^{-2r+2s} E_2^{5r-s}]_{4k+t} = 0$ for all $k \in \mathbb{N}_0$.

Lemma 2.4. *Let $r, s \in \mathbb{N}_0$ and $t \in \{1, 2, 3\}$. Suppose that*

$$\begin{aligned} z^r w^s &+ (-1)^t i^t \left(\frac{1}{2}(z+w) + \frac{1}{2}(z-w)i\right)^r \left(\frac{1}{2}(z+w) - \frac{1}{2}(z-w)i\right)^s \\ &+ (-1)^t w^r z^s + i^t \left(\frac{1}{2}(z+w) - \frac{1}{2}(z-w)i\right)^r \left(\frac{1}{2}(z+w) + \frac{1}{2}(z-w)i\right)^s = 0 \end{aligned} \tag{2.8}$$

holds for all complex numbers z and w. Then we have

$$[E_1^{-2r+2s} E_2^{5r-s}]_{4k+t} = 0 \quad \textit{for all } k \in \mathbb{N}_0.$$

Proof. Suppose that $r, s \in \mathbb{N}_0$ and $t \in \{1, 2, 3\}$ are such that (2.8) holds for all complex numbers z and w. We choose

$$z = A = \varphi(q), \quad w = B = \varphi(-q), \quad q \in \mathbb{C}, \ |q| < 1,$$

in (2.8). By Lemma 2.3 we have

$$\frac{1}{2}(z+w) + \frac{1}{2}(z-w)i = \left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)i = \varphi(iq),$$

$$\frac{1}{2}(z+w) - \frac{1}{2}(z-w)i = \left(\frac{A+B}{2}\right) - \left(\frac{A-B}{2}\right)i = \varphi(-iq),$$

so that (2.8) gives

$$\sum_{h=0}^{3} i^{3th} \varphi^r(i^h q) \varphi^s(-i^h q) = 0.$$

Then, by Lemma 2.2 with modulus 4, that is by (2.1), we obtain

$$[\varphi^r(q)\varphi^s(-q)]_{4k+t} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Appealing to (2.3), we deduce

$$[E_1^{-2r+2s}E_2^{5r-s}E_4^{-2r}]_{4k+t} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Finally, taking $f(q) = E_1^{-2r+2s}E_2^{5r-s}E_4^{-2r}$ and $g(q) = E_1^{2r}$ in Lemma 2.1, as $[f(q)]_{4k+t} = 0$ and $f(q)g(q^4) = E_1^{-2r+2s}E_2^{5r-s}$, we obtain

$$[E_1^{-2r+2s}E_2^{5r-s}]_{4k+t} = 0 \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

Our final lemma in this section gives the analogous result to Lemma 2.4 for modulus 8.

Lemma 2.5. *Let* $\omega := e^{2\pi i/8} = \frac{1+i}{\sqrt{2}}$. *Let* $r, s, t, u \in \mathbb{N}_0$ *and* $l \in \{1, 2, 3, 4, 5, 6, 7\}$. *For* $z, w \in \mathbb{C}$ *define*

$$J(\pm) := \left(\frac{z+w}{2}\right) \pm \left(\frac{z-w}{2}\right) i,$$

$$K(\pm) := \left(\frac{\left(\frac{z^2+w^2}{2}\right)^{1/2} + (zw)^{1/2}}{2}\right) \pm \left(\frac{\left(\frac{z^2+w^2}{2}\right)^{1/2} - (zw)^{1/2}}{2}\right) i,$$

$$L(\pm) := \left(\frac{zw(z^2+w^2)}{2}\right)^{1/4} \pm \left(\frac{z-w}{2}\right)\omega,$$

$$M(\pm) := \left(\frac{zw(z^2+w^2)}{2}\right)^{1/4} \pm \left(\frac{z-w}{2}\right)\omega i.$$

Next define

$$T_0 := z^r w^s \left(\frac{z^2+w^2}{2}\right)^{t/2} \left(\frac{z+w}{2}\right)^u,$$

$$T_1 := L(+)^r L(-)^s K(+)^t \left(\frac{zw(z^2+w^2)}{2}\right)^{u/4},$$

$$T_2 := J(+)^r J(-)^s (zw)^{t/2} \left(\frac{z+w}{2}\right)^u,$$

$$T_3 := M(+)^r M(-)^s K(-)^t \left(\frac{zw(z^2+w^2)}{2}\right)^{u/4},$$

$$T_4 := w^r z^s \left(\frac{z^2+w^2}{2}\right)^{t/2} \left(\frac{z+w}{2}\right)^u,$$

$$T_5 := L(-)^r L(+)^s K(+)^t \left(\frac{zw(z^2+w^2)}{2}\right)^{u/4},$$

$$T_6 := J(-)^r J(+)^s (zw)^{t/2} \left(\frac{z+w}{2}\right)^u,$$

$$T_7 := M(-)^r M(+)^s K(-)^t \left(\frac{zw(z^2+w^2)}{2}\right)^{u/4}.$$

Suppose that

$$\sum_{h=0}^{7} \omega^{7lh} T_h = 0 \tag{2.9}$$

holds for all complex numbers z and w. Then we have

$$[E_1^{-2r+2s} E_2^{5r-s-2t} E_4^{-2r+5t-2u}]_{8k+l} = 0 \quad \textit{for all } k \in \mathbb{N}_0.$$

Proof. Suppose that $r, s, t, u \in \mathbb{N}_0$ and $l \in \{1, 2, 3, 4, 5, 6, 7\}$ are such that (2.9) holds for all $z, w \in \mathbb{C}$. We choose

$$z = A = \varphi(q), \quad w = B = \varphi(-q), \quad q \in \mathbb{C}, \ |q| < 1.$$

By Lemma 2.3 we have

$$\begin{aligned}
\frac{z+w}{2} &= \varphi(q^4), \\
\left(\frac{z^2+w^2}{2}\right)^{1/2} &= \varphi(q^2), \\
(zw)^{1/2} &= \varphi(-q^2), \\
\left(\frac{zw(z^2+w^2)}{2}\right)^{1/4} &= \varphi(-q^4), \\
J(\pm) &= \varphi(\pm iq), \\
K(\pm) &= \varphi(\pm iq^2), \\
L(\pm) &= \varphi(\pm \omega q), \\
M(\pm) &= \varphi(\pm i\omega q),
\end{aligned}$$

and for $h = 0, 1, 2, \ldots, 7$

$$T_h = \varphi^r(\omega^h q)\varphi^s(-\omega^h q)\varphi^t(i^h q^2)\varphi^u((-1)^h q^4).$$

Hence (2.9) asserts that

$$\sum_{h=0}^{7} \omega^{7lh} \varphi^r(\omega^h q)\varphi^s(-\omega^h q)\varphi^t(i^h q^2)\varphi^u((-1)^h q^4) = 0$$

for all $q \in \mathbb{C}$, $|q| < 1$. Thus by (2.2) we have

$$[\varphi^r(q)\varphi^s(-q)\varphi^t(q^2)\varphi^u(q^4)]_{8k+l} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Appealing to (2.3) we deduce

$$[E_1^{-2r+2s}E_2^{5r-s-2t}E_4^{-2r+5t-2u}E_8^{-2t+5u}E_{16}^{-2u}]_{8k+l} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Finally, taking

$$f(q) = E_1^{-2r+2s}E_2^{5r-s-2t}E_4^{-2r+5t-2u}E_8^{-2t+5u}E_{16}^{-2u}$$

and

$$g(q) = E_1^{2t-5u}E_2^{2u}$$

in Lemma 2.1, as $[f(q)]_{8k+l} = 0$ and $f(q)g(q^8) = E_1^{-2r+2s}E_2^{5r-s-2t}E_4^{-2r+5t-2u}$, we obtain

$$[E_1^{-2r+2s}E_2^{5r-s-2t}E_4^{-2r+5t-2u}]_{8k+l} = 0 \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

3. Proof of Theorem 1.1

In this section, we determine for each $l \in \{1, 2, 3, 4\}$ all the pairs $(a, b) \in \mathbb{Z}^2$ such that

$$[E_1^a E_2^b]_{4k+l} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

This is accomplished in two steps. First we determine the candidate pairs (a, b) from the requirement that $[E_1^a E_2^b]_{4k+l} = 0$ for small values of k. Second we show that these candidate pairs (a, b) actually satisfy $[E_1^a E_2^b]_{4k+l} = 0$ for all $k \in \mathbb{N}_0$. If $a = 0$ this is deduced from the obvious identity $[E_2^b]_{2k+1} = 0$ for all $k \in \mathbb{N}_0$. If $a \neq 0$ then all candidate pairs turn out to satisfy $a + 2b \equiv 0 \pmod 8$ and we check that the identity (2.8) of Lemma 2.4 holds with $r = (a+2b)/8$ and $s = (5a+2b)/8$. This completes the proof of Theorem 1.1.

Lemma 3.1. *Let $a, b \in \mathbb{Z}$. If*

$$[E_1^a E_2^b]_{4k+1} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b) = (0, b).$$

Proof. Suppose $[E_1^a E_2^b]_{4k+1} = 0$ for all $k \in \mathbb{N}_0$. Then $[E_1^a E_2^b]_1 = 0$. But $[E_1^a E_2^b]_1 = -a$, so $a = 0$. $\square$

Lemma 3.2. *Let $a, b \in \mathbb{Z}$. If*

$$[E_1^a E_2^b]_{4k+2} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b) = (-4, 14),\ (-2, 5),\ (0, 0),\ (2, -1),\ (4, 2).$$

Proof. Suppose $[E_1^aE_2^b]_{4k+2} = 0$ for all $k \in \mathbb{N}_0$. Then using MAPLE we obtain

$$0 = [E_1^aE_2^b]_2 = -\frac{3}{2}a - b + \frac{1}{2}a^2 \tag{3.1}$$

and

$$0 = [E_1^aE_2^b]_6 = -2a - \frac{4}{3}b + \frac{1697}{360}a^2 + 4ab + \frac{3}{2}b^2 - \frac{55}{16}a^3 - \frac{77}{24}a^2b - \frac{3}{4}ab^2 - \frac{1}{6}b^3 + \frac{113}{144}a^4 + \frac{3}{4}a^3b + \frac{1}{4}a^2b^2 - \frac{1}{16}a^5 - \frac{1}{24}a^4b + \frac{1}{720}a^6. \tag{3.2}$$

From (3.1) we have

$$b = -\frac{3}{2}a + \frac{1}{2}a^2. \tag{3.3}$$

Using (3.3) in (3.2), we deduce

$$0 = \frac{64}{45}a^2 - \frac{4}{9}a^4 + \frac{1}{45}a^6. \tag{3.4}$$

Solving the sextic equation (3.4) for a, we obtain

$$a = -4, -2, 0, 2, 4.$$

The corresponding values of b are (from (3.3))

$$b = 14, 5, 0, -1, 2.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let $a, b \in \mathbb{Z}$. If*

$$[E_1^aE_2^b]_{4k+3} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b) = (-8, 24),\ (-4, 10),\ (-2, 5),\ (0, b),\ (2, -1),\ (4, -2),\ (8, 0).$$

Proof. Suppose $[E_1^aE_2^b]_{4k+3} = 0$ for all $k \in \mathbb{N}_0$. Then using MAPLE we obtain

$$0 = [E_1^aE_2^b]_3 = -\frac{4}{3}a + \frac{3}{2}a^2 + ab - \frac{1}{6}a^3 \tag{3.5}$$

and

$$0 = [E_1^aE_2^b]_7 = -\frac{8}{7}a + \frac{92}{15}a^2 + \frac{68}{15}ab - \frac{2021}{360}a^3 - 6a^2b - \frac{13}{6}ab^2 + \frac{89}{48}a^4 + \frac{49}{24}a^3b + \frac{3}{4}a^2b^2 + \frac{1}{6}ab^3 - \frac{35}{144}a^5 - \frac{1}{4}a^4b - \frac{1}{12}a^3b^2 + \frac{1}{80}a^6 + \frac{1}{120}a^5b - \frac{1}{5040}a^7. \tag{3.6}$$

Clearly $a = 0$ satisfies both of these equations. If $a \neq 0$, (3.5) gives

$$b = \frac{4}{3} - \frac{3}{2}a + \frac{1}{6}a^2. \tag{3.7}$$

Using (3.7) in (3.6), we obtain the sextic equation

$$\frac{4096}{2835} - \frac{64}{135}a^2 + \frac{4}{135}a^4 - \frac{1}{2835}a^6 = 0. \tag{3.8}$$

Solving (3.8), we find

$$a = -8, -4, -2, 2, 4, 8.$$

The corresponding values of b are (from (3.7))

$$b = 24, 10, 5, -1, -2, 0.$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let $a, b \in \mathbb{Z}$. If*

$$[E_1^a E_2^b]_{4k+4} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b) = (0, 0).$$

Proof. Suppose $[E_1^a E_2^b]_{4k+4} = 0$ for all $k \in \mathbb{N}_0$. Using MAPLE to solve the seven equations in a and b resulting from

$$0 = [E_1^a E_2^b]_{4k+4}, \quad k = 0, 1, 2, 3, 4, 5, 6,$$

we obtain $(a, b) = (0, 0)$. □

Proof of Theorem 1.1. (i) Let $b \in \mathbb{Z}$. In the power series expansion of E_2^b only even powers of q occur. Thus $[E_2^b]_{2k+1} = 0$ for all $k \in \mathbb{N}_0$. Hence, for $(a, b) = (0, b)$, we have

$$[E_1^a E_2^b]_{4k+1} = [E_2^b]_{4k+1} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Part (i) of Theorem 1.1 now follows by Lemma 3.1.

(ii) Using MAPLE we find that the identity (2.8) of Lemma 2.4 with $t = 2$ holds for

$$(r, s) = (3, 1),\ (1, 0),\ (0, 0),\ (0, 1),\ (1, 3)$$

so that by Lemma 2.4 we have $[E_1^a E_2^b]_{4k+2} = 0$ for

$$(a, b) = (-4, 14),\ (-2, 5),\ (0, 0),\ (2, -1),\ (4, 2),$$

respectively. Part (ii) of Theorem 1.1 now follows by Lemma 3.2.

(iii) Let $b \in \mathbb{Z}$. First we consider $a = 0$. From the proof of (i) we have $[E_2^b]_{2k+1} = 0$ for all $k \in \mathbb{N}_0$ so that for $(a, b) = (0, b)$, we have

$$[E_1^a E_2^b]_{4k+3} = [E_2^b]_{4k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Next we consider $a \neq 0$. Using MAPLE we find that the identity (2.8) of Lemma 2.4 with $t = 3$ holds for

$$(r, s) = (5, 1),\ (2, 0),\ (1, 0),\ (0, 1),\ (0, 2),\ (1, 5)$$

so that by Lemma 2.4 we have $[E_1^a E_2^b]_{4k+3} = 0$ for

$$(a, b) = (-8, 24),\ (-4, 10),\ (-2, 5),\ (2, -1),\ (4, -2),\ (8, 0),$$

respectively. Part (iii) of Theorem 1.1 now follows by Lemma 3.3.

(iv) Clearly for $(a, b) = (0, 0)$ we have

$$[E_1^a E_2^b]_{4k+4} = [1]_{4k+4} = 0 \quad \text{for all } k \in \mathbb{N}_0$$

and part (iv) of Theorem 1.1 now follows by Lemma 3.4.

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

For $l \in \{1, 2, 3, 4, 5\}$ we determine all $(a, b, c) \in \mathbb{Z}^3$ such that

$$[E_1^a E_2^b E_4^c]_{8k+l} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

For $l \in \{6, 7, 8\}$ we give some triples $(a, b, c) \in \mathbb{Z}^3$ such that $[E_1^a E_2^b E_4^c]_{8k+l} = 0$ for all $k \in \mathbb{N}_0$ but we cannot be sure that we have all of them as MAPLE was unable to solve the equations in a, b, c giving the candidate triples (a, b, c). We proceed as in the proof of Theorem 1.1 except that Lemma 2.5 is used instead of Lemma 2.4.

Lemma 4.1. *Let $a, b, c \in \mathbb{Z}$. If*

$$[E_1^a E_2^b E_4^c]_{8k+1} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b, c) = (0, b, c).$$

Proof. Suppose $[E_1^a E_2^b E_4^c]_{8k+1} = 0$ for all $k \in \mathbb{N}_0$. Then $[E_1^a E_2^b E_4^c]_1 = 0$. But $[E_1^a E_2^b E_4^c]_1 = -a$, so $a = 0$. □

Lemma 4.2. *Let $a, b, c \in \mathbb{Z}$. If*

$$[E_1^a E_2^b E_4^c]_{8k+2} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b, c) = (-6, 27, -6),\ (-4, 14, c),\ (-2, 5, c),\ (0, 0, c),\ (2, -1, c),\ (4, 2, c),\ (6, 9, 0).$$

Proof. Using MAPLE to solve the four equations

$$[E_1^a E_2^b E_4^c]_m = 0, \quad m = 2, 10, 18, 26,$$

in the three unknowns a, b, c, we obtain the seven solutions stated in the lemma. □

Lemma 4.3. *Let $a, b, c \in \mathbb{Z}$. If*

$$[E_1^a E_2^b E_4^c]_{8k+3} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b, c) = (-10, 33, -10),\ (-8, 24, c),\ (-4, 10, c),\ (-2, 5, c),$$
$$(0, b, c),\ (2, -1, c),\ (4, -2, c),\ (8, 0, c),\ (10, 3, 0).$$

Proof. Using MAPLE to solve the five equations

$$[E_1^a E_2^b E_4^c]_m = 0, \quad m = 3, 11, 19, 27, 35,$$

in the three unknowns a, b, c, we obtain the nine solutions stated in the lemma. □

Lemma 4.4. *Let $a, b, c \in \mathbb{Z}$. If*

$$[E_1^a E_2^b E_4^c]_{8k+4} = 0 \quad \textit{for all } k \in \mathbb{N}_0,$$

then

$$(a, b, c) = (-6, 15, 0),\ (-4, 10, 0),\ (-4, 12, -9),\ (-4, 14, -14),$$
$$(-4, 16, -15),\ (-4, 18, -12),\ (-2, 1, 14),\ (-2, 3, 5),$$
$$(-2, 5, 0),\ (-2, 7, -1),\ (-2, 9, 2),\ (0, -4, 14),\ (0, -2, 5),\ (0, 0, 0),$$
$$(0, 2, -1),\ (0, 4, 2),\ (2, -5, 16),\ (2, -3, 7),\ (2, -1, 2),$$
$$(2, 1, 1),\ (2, 3, 4),\ (4, -2, 4),\ (4, 0, -5),\ (4, 2, -10),$$
$$(4, 4, -11),\ (4, 6, -8),\ (6, -3, 6).$$

Proof. Using MAPLE to solve the four equations

$$[E_1^a E_2^b E_4^c]_m = 0, \quad m = 4, 12, 20, 28,$$

in the three unknowns a, b, c, we obtain the 27 solutions stated in the lemma. □

Lemma 4.5. *Let $a, b, c \in \mathbb{Z}$. If*

$$[E_1^a E_2^b E_4^c]_{8k+5} = 0 \quad \textit{for all } k \in \mathbb{N}_0$$

then

$$(a, b, c) = (-8, 20, -2),\ (-8, 22, -11),\ (-8, 24, -16),\ (-8, 26, -17),$$
$$(-8, 28, -14),\ (-6, 15, -2),\ (-4, 6, 12),\ (-4, 8, 3),$$
$$(-4, 10, -2),\ (-4, 12, -3),\ (-4, 14, 0),\ (-2, 1, 12),$$
$$(-2, 3, 3),\ (-2, 5, -2),\ (-2, 7, -3),\ (-2, 9, 0),\ (0, b, c),$$

$$(2,-5,14),\ (2,-3,5),\ (2,-1,0),\ (2,1,-1),$$
$$(2,3,2),\ (4,-6,16),\ (4,-4,7),\ (4,-2,2),$$
$$(4,0,1),\ (4,2,4),\ (6,-3,4),\ (8,-4,6),$$
$$(8,-2,-3),\ (8,0,-8),\ (8,2,-9),\ (8,4,-6).$$

Proof. Using MAPLE to solve the three equations

$$[E_1^a E_2^b E_4^c]_m = 0, \quad m = 5, 13, 21,$$

in the three unknowns a, b, c, we obtain the 33 solutions stated in the lemma. □

In searching for the solutions $(a,b,c) \in \mathbb{Z}^3$ of $[E_1^a E_2^b E_4^c]_{8k+6} = 0$, $[E_1^a E_2^b E_4^c]_{8k+7} = 0$ and $[E_1^a E_2^b E_4^c]_{8k+8} = 0$ for small values of k, MAPLE was unable to find all the solutions.

Proof of Theorem 1.2. (i) Let $b, c \in \mathbb{Z}$. In the power series expansion of $E_2^b E_4^c$ only even powers of q occur. Thus $[E_2^b E_4^c]_{2k+1} = 0$ for all $k \in \mathbb{N}_0$. Hence, for $(a,b,c) = (0,b,c)$, we have

$$[E_1^a E_2^b E_4^c]_{8k+1} = [E_2^b E_4^c]_{8k+1} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Part (i) of Theorem 1.2 now follows by Lemma 4.1.

(ii) Let

$$(a,b) = (-4,14),\ (-2,5),\ (0,0),\ (2,-1),\ (4,2)$$

and $c \in \mathbb{Z}$. Then, by Theorem 1.1(ii), we have

$$[E_1^a E_2^b]_{4k+2} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Hence, by Lemma 2.1, we have

$$[E_1^a E_2^b E_4^c]_{8k+2} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Using MAPLE we find that the identity (2.9) of Lemma 2.5 with $l = 2$ is satisfied for $(r,s,t,u) = (7,4,2,1)$ and $(4,7,2,1)$ so that by Lemma 2.5, we have

$$[E_1^{-6} E_2^{27} E_4^{-6}]_{8k+2} = 0, \quad [E_1^6 E_2^9 E_4^0]_{8k+2} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Part (ii) of Theorem 1.2 now follows by Lemma 4.2.

(iii) Let

$$(a,b) = (-8,24),\ (-4,10),\ (-2,5),\ (0,b),\ (2,-1),\ (4,-2),\ (8,0),$$

where $b \in \mathbb{Z}$. By Theorem 1.1(iii) we have

$$[E_1^a E_2^b]_{4k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Hence, by Lemma 2.1, we have for $c \in \mathbb{Z}$

$$[E_1^a E_2^b E_4^c]_{4k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Table 1. Values of (a,b,c) for which $[E_1^aE_2^bE_4^c]_{8k+4} = 0$.

No.	r	s	t	u	a	b	c
1	4	1	2	1	−6	15	0
2	3	1	2	2	−4	10	0
3	3	1	1	4	−4	12	−9
4	3	1	0	4	−4	14	−14
5	4	2	1	6	−4	16	−15
6	4	2	0	2	−4	18	−12
7	2	1	4	1	−2	1	14
8	2	1	3	3	−2	3	5
9	2	1	2	3	−2	5	0
10	2	1	1	1	−2	7	−1
11	3	2	2	1	−2	9	2
12	1	1	4	2	0	−4	14
13	0	0	1	0	0	−2	5
14	0	0	0	0	0	0	0
15	1	1	1	2	0	2	−1
16	2	2	2	2	0	4	2
17	1	2	4	1	2	−5	16
18	1	2	3	3	2	−3	7
19	1	2	2	3	2	−1	2
20	1	2	1	1	2	1	1
21	2	3	2	1	2	3	4
22	1	3	2	2	4	−2	4
23	1	3	1	4	4	0	−5
24	1	3	0	4	4	2	−10
25	2	4	1	6	4	4	−11
26	2	4	0	2	4	6	−8
27	1	4	2	1	6	−3	6

Thus

$$[E_1^aE_2^bE_4^c]_{8k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Using MAPLE we find that the identity (2.9) of Lemma 2.5 with $l = 3$ is satisfied for $(r,s,t,u) = (8,3,2,2)$ and $(3,8,2,2)$. Hence, by Lemma 2.5, we have

$$[E_1^{-10}E_2^{33}E_4^{-10}]_{8k+3} = 0, \quad [E_1^{10}E_2^{3}E_4^{0}]_{8k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Part (iii) of Theorem 1.2 now follows by Lemma 4.3.

(iv) Using MAPLE we find that the identity (2.9) of Lemma 2.5 with $l = 4$ is satisfied for the 27 values of (r,s,t,u) listed in Table 1. The corresponding 27 values of (a,b,c) for which $[E_1^aE_2^bE_4^c]_{8k+4} = 0$, for all $k \in \mathbb{N}_0$, are also given.

Part (iv) of Theorem 1.2 now follows by Lemma 4.4.

(v) Let $b, c \in \mathbb{Z}$. From the proof of part (i) we have for $(a,b,c) = (0,b,c)$

$$[E_1^aE_2^bE_4^c]_{8k+5} = [E_2^bE_4^c]_{8k+5} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Table 2. Values of (a,b,c) for which $[E_1^aE_2^bE_4^c]_{8k+5}=0$.

No.	r	s	t	u	a	b	c
1	5	1	2	1	−8	20	−2
2	5	1	1	3	−8	22	−11
3	5	1	0	3	−8	24	−16
4	6	2	1	5	−8	26	−17
5	6	2	0	1	−8	28	−14
6	4	1	2	2	−6	15	−2
7	3	1	4	1	−4	6	12
8	3	1	3	3	−4	8	3
9	3	1	2	3	−4	10	−2
10	3	1	1	1	−4	12	−3
11	4	2	2	1	−4	14	0
12	2	1	4	2	−2	1	12
13	1	0	1	0	−2	3	3
14	1	0	0	0	−2	5	−2
15	2	1	1	2	−2	7	−3
16	3	2	2	2	−2	9	0
17	1	2	4	2	2	−5	14
18	1	2	3	4	2	−3	5
19	0	1	0	0	2	−1	0
20	1	2	1	2	2	1	−1
21	2	3	2	2	2	3	2
22	1	3	4	1	4	−6	16
23	1	3	3	3	4	−4	7
24	1	3	2	3	4	−2	2
25	1	3	1	1	4	0	1
26	2	4	2	1	4	2	4
27	1	4	2	2	6	−3	4
28	1	5	2	1	8	−4	6
29	1	5	1	3	8	−2	−3
30	1	5	0	3	8	0	−8
31	2	6	1	5	8	2	−9
32	2	6	0	1	8	4	−6

Using MAPLE we find that the identity (2.9) of Lemma 2.5 with $l=5$ is satisfied for the 32 values of (r,s,t,u) listed in Table 2. The corresponding 32 values of (a,b,c) for which $[E_1^aE_2^bE_4^c]_{8k+5}=0$, for all $k\in\mathbb{N}_0$, are also given.
Part (v) of Theorem 1.2 now follows by Lemma 4.5.

(vi) Using MAPLE we find that the identity (2.9) of Lemma 2.5 with $l=6$ is satisfied for the 44 values of (r,s,t,u) listed in Table 3. The corresponding 44 values of (a,b,c) for which $[E_1^aE_2^bE_4^c]_{8k+6}=0$, for all $k\in\mathbb{N}_0$, are also given.

Part (vi) of Theorem 1.2 now follows.

(vii) Let

$$(a,b)=(-8,24),\ (-4,10),\ (-2,5),\ (0,b),\ (2,-1),\ (4,-2),\ (8,0),$$

Table 3. Values of (a,b,c) for which $[E_1^a E_2^b E_4^c]_{8k+6} = 0$.

No.	r	s	t	u	a	b	c
1	7	1	0	2	−12	34	−18
2	6	1	2	1	−10	25	−4
3	9	4	4	1	−10	33	0
4	5	1	2	2	−8	20	−4
5	4	1	4	1	−6	11	10
6	4	1	2	3	−6	15	−4
7	5	2	2	1	−6	19	−2
8	3	1	4	2	−4	6	10
9	2	0	0	0	−4	10	−4
10	3	1	1	4	−4	12	−9
11	4	2	1	6	−4	16	−15
12	4	2	0	4	−4	18	−16
13	5	3	0	2	−4	22	−14
14	2	1	6	1	−2	−3	24
15	2	1	4	3	−2	1	10
16	2	1	3	3	−2	3	5
17	2	1	1	1	−2	7	−1
18	3	2	2	3	−2	9	−2
19	4	3	2	1	−2	13	0
20	1	1	6	2	0	−8	24
21	0	0	2	0	0	−4	10
22	0	0	1	0	0	−2	5
23	1	1	1	2	0	2	−1
24	1	1	0	0	0	4	−2
25	3	3	2	2	0	8	0
26	1	2	6	1	2	−9	26
27	1	2	4	3	2	−5	12
28	1	2	3	3	2	−3	7
29	1	2	1	1	2	1	1
30	2	3	2	3	2	3	0
31	3	4	2	1	2	7	2
32	1	3	4	2	4	−6	14
33	0	2	0	0	4	−2	0
34	1	3	1	4	4	0	−5
35	2	4	1	6	4	4	−11
36	2	4	0	4	4	6	−12
37	3	5	0	2	4	10	−10
38	1	4	4	1	6	−7	16
39	1	4	2	3	6	−3	2
40	2	5	2	1	6	1	4
41	1	5	2	2	8	−4	4
42	1	6	2	1	10	−5	6
43	4	9	4	1	10	3	10
44	1	7	0	2	12	−2	−6

where $b \in \mathbb{Z}$. By Theorem 1.1(iii) we have

$$[E_1^a E_2^b]_{4k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Hence, by Lemma 2.1, we have for $c \in \mathbb{Z}$

$$[E_1^a E_2^b E_4^c]_{4k+3} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Thus

$$[E_1^a E_2^b E_4^c]_{8k+7} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

As we have already mentioned MAPLE was unable to solve the equations in a, b, c resulting from $[E_1^a E_2^b E_4^c]_{8k+7} = 0$ for several small values of $k \in \mathbb{N}_0$. Thus instead we ran through $(r, s, t, u) \in \mathbb{N}_0^4$ satisfying

$$0 \le r, s, t, u \le 9$$

and determined which quadruples (r, s, t, u) satisfied the identity (2.9) of Lemma 2.5. The computer found 50 quadruples (r, s, t, u) that satisfied (2.9). They are listed in Table 4 along with the corresponding values of (a, b, c).

Table 4. Values of (a, b, c) for which $[E_1^a E_2^b E_4^c]_{8k+7} = 0$.

No.	r	s	t	u	a	b	c
1	9	1	0	1	−16	44	−20
2	7	1	2	1	−12	30	−6
3	6	1	2	2	−10	25	−6
4	5	1	4	1	−8	16	8
5	5	1	2	3	−8	20	−6
6	5	1	1	3	−8	22	−11
7	6	2	1	5	−8	26	−17
8	6	2	0	3	−8	28	−18
9	7	3	0	1	−8	32	−16
10	4	1	4	2	−6	11	8
11	3	0	0	0	−6	15	−6
12	5	2	2	2	−6	19	−4
13	8	5	4	2	−6	27	0
14	3	1	6	1	−4	2	22
15	3	1	4	3	−4	6	8
16	3	1	3	3	−4	8	3
17	3	1	1	1	−4	12	−3
18	4	2	2	3	−4	14	−4
19	5	3	2	1	−4	18	−2
20	2	1	6	2	−2	−3	22
21	1	0	2	0	−2	1	8
22	1	0	1	0	−2	3	3
23	2	1	1	2	−2	7	−3
24	2	1	0	0	−2	9	−4
25	5	4	4	6	−2	13	−2

(*Continued*)

Table 4. (*Continued*)

No.	r	s	t	u	a	b	c
26	1	2	6	2	2	−9	24
27	0	1	2	0	2	−5	10
28	0	1	1	0	2	−3	5
29	1	2	1	2	2	1	−1
30	1	2	0	0	2	3	−2
31	3	4	2	2	2	7	0
32	1	3	6	1	4	−10	26
33	1	3	4	3	4	−6	12
34	1	3	3	3	4	−4	7
35	1	3	1	1	4	0	1
36	2	4	2	3	4	2	0
37	3	5	2	1	4	6	2
38	1	4	4	2	6	−7	14
39	0	3	0	0	6	−3	0
40	2	5	2	2	6	1	2
41	5	8	4	2	6	9	6
42	1	5	4	1	8	−8	16
43	1	5	2	3	8	−4	2
44	1	5	1	3	8	−2	−3
45	2	6	1	5	8	2	−9
46	2	6	0	3	8	4	−10
47	3	7	0	1	8	8	−8
48	1	6	2	2	10	−5	4
49	1	7	2	1	12	−6	6
50	1	9	0	1	16	−4	−4

Part (vii) of Theorem 1.2 now follows.

(viii) This part is trivial.

This completes the proof of Theorem 1.2. □

It is possible that the lists of parts (vi), (vii) and (viii) of Theorem 1.2 are complete but we cannot prove this.

5. Final Remarks

In principle the methods of this paper can be used to find the quadruples $(a,b,c,d) \in \mathbb{Z}^4$ such that

$$[E_1^a E_2^b E_4^c E_8^d]_{16k+l} = 0 \quad \text{for all } k \in \mathbb{N}_0, \tag{5.1}$$

where $l \in \{1,2,\ldots,15\}$, but the details would be quite challenging! It uses

$$\varphi^r(q)\varphi^s(-q)\varphi^t(q^2)\varphi^u(q^4)\varphi^v(q^8)$$
$$= E_1^{-2r+2s} E_2^{5r-s-2t} E_4^{-2r+5t-2u} E_8^{-2t+5u-2v} E_{16}^{-2u+5v} E_{32}^{-2v}$$

and

$$[E_1^{-2r+2s}E_2^{5r-s-2t}E_4^{-2r+5t-2u}E_8^{-2t+5u-2v}E_{16}^{-2u+5v}E_{32}^{-2v}]_{16k+l}$$
$$= [E_1^{-2r+2s}E_2^{5r-s-2t}E_4^{-2r+5t-2u}E_8^{-2t+5u-2v}]_{16k+l},$$

and requires the parametrizations in terms of A and B of $\varphi(\pm\omega^h q)$, $\varphi(\omega^{2h}q^2)$, $\varphi(\omega^{4h}q^4)$ and $\varphi(\omega^{8h}q^8)$, where

$$\omega = e^{\frac{2\pi i}{16}} = \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}$$

and $h \in \{0, 1, 2, \ldots, 15\}$. There are identities of the type (5.1) to be discovered. We give just one such example.

Theorem 5.1.

$$[E_1^4E_4^9E_8^{-2}]_{16k+13} = 0 \quad \textit{for all } k \in \mathbb{N}_0.$$

Proof. We prove this result by appealing to a recent theorem of the fourth author [2, Theorem 1.1, p. 80]. We take

$$r = t = u = v = w = x = 0, \quad s = y = 1,$$

in this theorem to obtain

$$[qE_1^4E_4^9E_8^{-2}]_n = \sum_{\substack{(x_1, x_2, x_3) \in \mathbb{Z}^3 \\ x_1^2 + x_2^2 + 2x_3^2 = n}} (x_1^4 - 3x_1^2x_2^2),$$

which is valid for all $n \in \mathbb{N}$. As $x_1^2 + x_2^2 + 2x_3^2 \not\equiv 14 \pmod{16}$ we deduce that

$$[qE_1^4E_4^9E_8^{-2}]_{16k+14} = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

The asserted result follows on dividing by q. □

Further examples of this type are given in [3], as well as references to related results in the literature.

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