Historical Remark on Ramanujan's Tau Function

Kenneth S. Williams

Abstract. It is shown that Ramanujan could have proved a special case of his conjecture that his tau function is multiplicative without any recourse to modularity results.

1. INTRODUCTION. In his path-breaking paper on arithmetic functions published in 1916, Ramanujan [**6**, eq. (92)] introduced the function $\tau(n)$ that in his honor is now called the Ramanujan tau function. This function is defined for all positive integers n by

$$q\prod_{m=1}^{\infty}(1-q^m)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n, \quad q\in\mathbb{C}, \quad |q|<1. \tag{1}$$

Ramanujan calculated the first 30 values of $\tau(n)$ [**6**, Table V] and observed that $\tau(n)$ appeared to be multiplicative [**6**, eq. (103)], that is,

$$\tau(n_1n_2)=\tau(n_1)\tau(n_2), \quad n_1,n_2\in\mathbb{N}, \quad \gcd(n_1,n_2)=1. \tag{2}$$

This was proved by Mordell [**5**] shortly afterward using modular techniques, which were unknown to Ramanujan. A modern proof of (2) is given, for example, in [**4**, p. 298]. The author is not aware of any proof of (2) that does not appeal to the theory of modular forms.

It is known from the theory of modular forms for all primes p and all positive integers n that the following property of $\tau(n)$ holds, namely,

$$\tau(pn)=\tau(p)\tau(n)-p^{11}\tau(n/p), \tag{3}$$

where $\tau(n/p)=0$ if p does not divide n [**4**, p. 298]. Moreover, a simple induction argument using (3) gives the multiplicativity property (2) of $\tau(n)$; see, for example, [**4**, Cor. 5.6, p. 298].

The purpose of this historical note is to show that Ramanujan had all the tools necessary to prove the special case of (3) when $p=2$, namely,

$$\tau(2n)=\tau(2)\tau(n)-2^{11}\tau(n/2), \quad n\in\mathbb{N}, \tag{4}$$

from which the multiplicativity property

$$\tau(2^kN)=\tau(2^k)\tau(N), \quad \text{for } k\in\mathbb{N}\cup\{0\},\ N\in\mathbb{N},\ \text{and } N \text{ odd}, \tag{5}$$

follows.

http://dx.doi.org/10.4169/amer.math.monthly.122.01.30
MSC: Primary 11A05, Secondary 11F20; 11F27; 01A60; 11F11

2. PROOF OF (4) IN THE SPIRIT OF RAMANUJAN. Ramanujan defined a general theta function [**3**, Definition 1.2.1, p. 6] and studied its properties. An important special case of his function is the theta function $\varphi(q)$ given by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \text{for } q \in \mathbb{C} \text{ and } |q| < 1. \tag{6}$$

Ramanujan knew many properties of $\varphi(q)$, including the two simple identities

$$\varphi^2(q^2) = \frac{1}{2}(\varphi^2(q) + \varphi^2(-q)) \quad \text{and} \quad \varphi^2(-q^2) = \varphi(q)\varphi(-q); \tag{7}$$

see [**3**, pp. 15, 72]. Ramanujan also used extensively three Eisenstein series, which he denoted by $P(q)$, $Q(q)$, and $R(q)$ [**6**, eq. (25)], the latter two of which are

$$Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad q \in \mathbb{C}, \ \ |q| < 1, \tag{8}$$

and

$$R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}, \quad q \in \mathbb{C}, \ \ |q| < 1. \tag{9}$$

Ramanujan's pioneering work established the relationship between Eisenstein series and theta functions. He was well aware of results of the type

$$Q(q) = (1 + 14x + x^2)z^4 \tag{10}$$

and

$$R(q) = (1 - 33x - 33x^2 + x^3)z^6, \tag{11}$$

[**3**, Theorems 5.4.11 and 5.4.12], where [**3**, p. 120]

$$x = x(q) := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)} \quad \text{and} \quad z = z(q) := \varphi^2(q), \tag{12}$$

so that

$$Q(q) = 16\varphi^8(q) - 16\varphi^4(q)\varphi^4(-q) + \varphi^8(-q) \tag{13}$$

and

$$R(q) = -64\varphi^{12}(q) + 96\varphi^8(q)\varphi^4(-q) - 30\varphi^4(q)\varphi^8(-q) - \varphi^{12}(-q). \tag{14}$$

In [**6**, eq. (44)], Ramanujan proved, using only an elementary argument, the fundamental relation

$$1728q \prod_{m=1}^{\infty} (1-q^m)^{24} = Q^3(q) - R^2(q). \tag{15}$$

The following relation follows from (1) and (13)–(15):

$$16\sum_{n=1}^{\infty}\tau(n)q^n = \varphi^8(q)\varphi^{16}(-q) - \varphi^4(q)\varphi^{20}(-q). \tag{16}$$

Replacing q by $-q$ in (16) and adding the resulting equation to (16) gives

$$32\sum_{n=1}^{\infty}\tau(2n)q^{2n} = -\varphi^{20}(q)\varphi^4(-q) + \varphi^{16}(q)\varphi^8(-q) + \varphi^8(q)\varphi^{16}(-q) - \varphi^4(q)\varphi^{20}(-q). \tag{17}$$

Replacing q by q^2 in (16) and making use of (7) gives

$$256\sum_{n=1}^{\infty}\tau(n)q^{2n} = \varphi^{16}(q)\varphi^8(-q) - 2\varphi^{12}(q)\varphi^{12}(-q) + \varphi^8(q)\varphi^{16}(-q). \tag{18}$$

Replacing q by q^2 in (18) and appealing to (7) gives

$$65536\sum_{n=1}^{\infty}\tau(n)q^{4n} = \varphi^{20}(q)\varphi^4(-q) - 4\varphi^{16}(q)\varphi^8(-q) + 6\varphi^{12}(q)\varphi^{12}(-q) - 4\varphi^8(q)\varphi^{16}(-q) + \varphi^4(q)\varphi^{20}(-q). \tag{19}$$

Then (17)–(19) give

$$\sum_{n=1}^{\infty}\tau(2n)q^{2n} + 24\sum_{n=1}^{\infty}\tau(n)q^{2n} + 2048\sum_{n=1}^{\infty}\tau(n)q^{4n} = 0 \tag{20}$$

so that

$$\tau(2n) + 24\tau(n) + 2048\tau(n/2) = 0, \quad n \in \mathbb{N}, \tag{21}$$

from which (4) follows as $\tau(2) = -24$ and $2^{11} = 2048$. ■

It would be very interesting to know if Ramanujan had a proof along these lines for the special case (5) of his conjecture.

3. PROOF OF (3) FOR $p = 3$. The question naturally arises, "Can (3) be proved for primes $p \neq 2$ in a similar manner to the elementary proof given in Section 2 for $p = 2$?" However, this seems to be quite difficult. We carry out the proof for the prime $p = 3$ and at the end of the proof summarize the difficulties involved in giving such a proof for an arbitrary prime $p > 3$.

As $\tau(3) = 252$ and $3^{11} = 177147$, we must prove analogously to (20) that

$$\sum_{n=1}^{\infty}\tau(3n)q^{3n} - 252\sum_{n=1}^{\infty}\tau(n)q^{3n} + 177147\sum_{n=1}^{\infty}\tau(n)q^{9n} = 0. \tag{22}$$

We determine the sum of each of the three infinite series in (22) individually in terms of the function φ. First, from (16), we have

$$16\sum_{n=1}^{\infty}\tau(n)q^{3n} = \varphi^8(q^3)\varphi^{16}(-q^3) - \varphi^4(q^3)\varphi^{20}(-q^3) \tag{23}$$

and

$$16\sum_{n=1}^{\infty}\tau(n)q^{9n} = \varphi^8(q^9)\varphi^{16}(-q^9) - \varphi^4(q^9)\varphi^{20}(-q^9). \tag{24}$$

Now we turn to $\sum_{n=1}^{\infty}\tau(3n)q^{3n}$. We let $\omega = \exp(2\pi i/3)$ and note that

$$1 + \omega^n + \omega^{2n} = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 0 & \text{if } n \not\equiv 0 \pmod 3. \end{cases} \tag{25}$$

By (25), we have

$$\begin{aligned} 48\sum_{n=1}^{\infty}\tau(3n)q^{3n} &= 16\sum_{n=1}^{\infty}\tau(n)(1+\omega^n+\omega^{2n})q^n \\ &= 16\sum_{n=1}^{\infty}\tau(n)q^n + 16\sum_{n=1}^{\infty}\tau(n)(\omega q)^n + 16\sum_{n=1}^{\infty}\tau(n)(\omega^2 q)^n, \end{aligned}$$

so that by (16),

$$\begin{aligned} 48\sum_{n=1}^{\infty}\tau(3n)q^{3n} = \varphi^8(q)\varphi^{16}(-q) &- \varphi^4(q)\varphi^{20}(-q) \\ + \varphi^8(\omega q)\varphi^{16}(-\omega q) &- \varphi^4(\omega q)\varphi^{20}(-\omega q) \\ + \varphi^8(\omega^2 q)\varphi^{16}(-\omega^2 q) &- \varphi^4(\omega^2 q)\varphi^{20}(-\omega^2 q). \end{aligned} \tag{26}$$

Next, we consider $\varphi(\omega q)$. From (6), we deduce that

$$\varphi(\omega q) = \sum_{n=-\infty}^{\infty}\omega^{n^2}q^{n^2} = \sum_{\substack{n=-\infty \\ n\equiv 0 \pmod 3}}^{\infty} q^{n^2} + \omega \sum_{\substack{n=-\infty \\ n\not\equiv 0 \pmod 3}}^{\infty} q^{n^2},$$

as $n^2 \equiv 1 \pmod 3$ for $n \not\equiv 0 \pmod 3$. Hence,

$$\varphi(\omega q) = \varphi(q^9) + \omega(\varphi(q) - \varphi(q^9)). \tag{27}$$

Similarly, we have

$$\varphi(-\omega q) = \varphi(-q^9) + \omega(\varphi(-q) - \varphi(-q^9)), \tag{28}$$

$$\varphi(\omega^2 q) = \varphi(q^9) + \omega^2(\varphi(q) - \varphi(q^9)), \tag{29}$$

and

$$\varphi(-\omega^2 q) = \varphi(-q^9) + \omega^2(\varphi(-q) - \varphi(-q^9)). \tag{30}$$

Using (27)–(30) in (26), we deduce

$$
\begin{aligned}
48 &\sum_{n=1}^{\infty} \tau(3n) q^{3n} \\
&= \varphi^8(q)\varphi^{16}(-q) - \varphi^4(q)\varphi^{20}(-q) \\
&\quad + (\varphi(q^9) + \omega(\varphi(q) - \varphi(q^9)))^8 (\varphi(-q^9) + \omega(\varphi(-q) - \varphi(-q^9)))^{16} \\
&\quad - (\varphi(q^9) + \omega(\varphi(q) - \varphi(q^9)))^4 (\varphi(-q^9) + \omega(\varphi(-q) - \varphi(-q^9)))^{20} \\
&\quad + (\varphi(q^9) + \omega^2(\varphi(q) - \varphi(q^9)))^8 (\varphi(-q^9) + \omega^2(\varphi(-q) - \varphi(-q^9)))^{16} \\
&\quad - (\varphi(q^9) + \omega^2(\varphi(q) - \varphi(q^9)))^4 (\varphi(-q^9) + \omega^2(\varphi(-q) - \varphi(-q^9)))^{20}. \qquad (31)
\end{aligned}
$$

Multiplying (22) by 48, and appealing to (31), (23), and (24), we must prove the following identity relating $\varphi(q)$, $\varphi(-q)$, $\varphi(q^3)$, $\varphi(-q^3)$, $\varphi(q^9)$, and $\varphi(-q^9)$, namely,

$$
\begin{aligned}
&\varphi^8(q)\varphi^{16}(-q) - \varphi^4(q)\varphi^{20}(-q) \\
&\quad + (\varphi(q^9) + \omega(\varphi(q) - \varphi(q^9)))^8 (\varphi(-q^9) + \omega(\varphi(-q) - \varphi(-q^9)))^{16} \\
&\quad - (\varphi(q^9) + \omega(\varphi(q) - \varphi(q^9)))^4 (\varphi(-q^9) + \omega(\varphi(-q) - \varphi(-q^9)))^{20} \\
&\quad + (\varphi(q^9) + \omega^2(\varphi(q) - \varphi(q^9)))^8 (\varphi(-q^9) + \omega^2(\varphi(-q) - \varphi(-q^9)))^{16} \\
&\quad - (\varphi(q^9) + \omega^2(\varphi(q) - \varphi(q^9)))^4 (\varphi(-q^9) + \omega^2(\varphi(-q) - \varphi(-q^9)))^{20} \\
&\quad - 756(\varphi^8(q^3)\varphi^{16}(-q^3) - \varphi^4(q^3)\varphi^{20}(-q^3)) \\
&\quad + 531441(\varphi^8(q^9)\varphi^{16}(-q^9) - \varphi^4(q^9)\varphi^{20}(-q^9)) = 0. \qquad (32)
\end{aligned}
$$

The most elementary way of proving the identity (32) known to the author is to use the (p, k)-parametrizations of $\varphi(q)$, $\varphi(q^3)$, $\varphi(q^9)$, $\varphi(-q)$, $\varphi(-q^3)$, and $\varphi(-q^9)$ due to Alaca, Alaca, and Williams [**2**]. (We emphasize that here p is a function of q and is not being used to denote a prime.) We note that all of these parametrizations have been proved without the use of modular forms. As in [**2**, p. 178], we set

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \quad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}, \qquad (33)$$

so that

$$\varphi(q) = (1 + 2p)^{3/4} k^{1/2} \qquad (34)$$

and

$$\varphi(q^3) = (1 + 2p)^{1/4} k^{1/2}. \qquad (35)$$

A. Alaca [**1**, Theorem 2.2, p. 156] has shown that

$$\varphi(q^9) = \frac{1}{3}(1 + 2p)^{3/4} k^{1/2} + \frac{2^{2/3}}{3}(1 + 2p)^{1/12}(1 - p)^{1/3}(2 + p)^{1/3} k^{1/2}. \qquad (36)$$

The "change of sign" principle [**2**, Theorem 11, p. 180] asserts that

$$p(-q) = \frac{-p}{1 + p} \quad \text{and} \quad k(-q) = (1 + p)^2 k. \qquad (37)$$

Changing q to $-q$ in (34)–(36) and appealing to (37), we obtain

$$\varphi(-q) = (1-p)^{3/4}(1+p)^{1/4}k^{1/2}, \tag{38}$$

$$\varphi(-q^3) = (1-p)^{1/4}(1+p)^{3/4}k^{1/2}, \tag{39}$$

and

$$\varphi(-q^9) = \frac{1}{3}(1-p)^{3/4}(1+p)^{1/4}k^{1/2} + \frac{2^{2/3}}{3}(1-p)^{1/12}(1+2p)^{1/3}(2+p)^{1/3}(1+p)^{1/4}k^{1/2}. \tag{40}$$

Using MAPLE to substitute (34)–(36) and (38)–(40) into (32) and to simplify the resulting expression, we find that it is equal to 0, thereby establishing the identity (32) and proving (3) in the case $p = 3$. ■

In attempting to extend this elementary argument to an arbitrary prime $p > 3$, three obstacles become apparent. First, we do not know at the outset the value of $\tau(p)$ to use in the analogue of (22). Secondly, we need to determine $\varphi(\omega q)$, where $\omega = \exp(2\pi i/p)$, analogously to (27). Finally, a parametrization of $\varphi(q)$, $\varphi(q^p)$, $\varphi(q^{p^2})$, $\varphi(-q)$, $\varphi(-q^p)$, and $\varphi(-q^{p^2})$ would be helpful in order to verify the identity analogous to (32) for a prime $p > 3$.

Ramanujan's tau function is nearly a century old. It is hoped that this historical note will encourage the reader to learn more of its interesting properties and its place in the theory of modular forms.

ACKNOWLEDGMENT. The author thanks the two anonymous referees for their useful suggestions.

REFERENCES

1. A. Alaca, Representations by quaternary quadratic forms whose coefficients are 1, 3 and 9, *Acta Arith.* **136** (2009) 151–166.
2. A. Alaca, Ş. Alaca, K. S. Williams, On the two-dimensional theta functions of the Borweins, *Acta Arith.* **124** (2006) 177–195.
3. B. C. Berndt, *Number Theory in the Spirit of Ramanujan.* American Mathematical Society, Providence, RI, 2006.
4. H. M. Farkas, I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group.* American Mathematical Society, Providence, RI, 2001.
5. L. J. Mordell, On Mr. Ramanujan's empirical expansions of modular functions, *Proc. Cambridge Philos. Soc.* **19** (1917) 117–124.
6. S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Philos. Soc.* **22** (1916) 159–184. [*Collected Papers of Srinivasa Ramanujan*, AMS Chelsea Publishing, American Mathematical Society, Providence, RI, 2000, pp. 136–162.]

KENNETH S. WILLIAMS received his B.Sc. from the University of Birmingham, England in 1962 and his M.A. and Ph.D. degrees from the University of Toronto in 1963 and 1965, respectively. In 1979, he was awarded a D.Sc. by the University of Birmingham for his research in the theory of numbers. He retired from Carleton University in 2002 after 36 years of service. In retirement, he enjoys time with his family, gardening, birding with his wife, and visiting with his young granddaughter Isabelle.
School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6
kwilliam@connect.carleton.ca