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Ternary quadratic forms $ax^2 + by^2 + cz^2$ representing all positive integers $8k + 4$

by

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1. Introduction. Let $\mathbb{N}$ denote the set of positive integers. Dickson [D2, Theorems V, X and VII] showed that the three ternary quadratic forms

$$x^2 + y^2 + 2z^2, \quad x^2 + 2y^2 + 3z^2, \quad x^2 + 2y^2 + 4z^2 \tag{1.1}$$

represent all positive integers $n \equiv 1 \pmod 2$, and Kaplansky [K, pp. 212–213] proved that there are no other such ternary forms with this property (see also Panaitopol [P, Theorem 1]).

Williams [W] has shown that the only ternary quadratic forms $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}$, $a \le b \le c$) representing all positive integers $n \equiv 2 \pmod 4$ are the nine forms

$$\begin{array}{lll} x^2 + y^2 + z^2, & x^2 + y^2 + 4z^2, & x^2 + y^2 + 5z^2, \\ x^2 + 2y^2 + 2z^2, & x^2 + 2y^2 + 6z^2, & x^2 + 2y^2 + 8z^2, \\ 2x^2 + 2y^2 + 4z^2, & 2x^2 + 4y^2 + 6z^2, & 2x^2 + 4y^2 + 8z^2. \end{array} \tag{1.2}$$

In this note, subject to the validity of the following conjecture, we determine all ternary forms $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}$, $a \le b \le c$) that represent all positive integers $n \equiv 4 \pmod 8$.

Conjecture 1.1. *The ternary quadratic form $x^2 + 2y^2 + 5z^2 + xz$ represents all odd positive integers.*

The form $x^2 + 2y^2 + 5z^2 + xz$ was given by Kaplansky [K, no. 20, p. 213] as a candidate for representing all odd positive integers. Elkies (see [K, p. 209]) has checked that this form represents all odd positive integers up to 16383. Rouse in his work on quadratic forms representing all odd positive integers formulated the conjecture that Kaplansky's form $x^2 + 2y^2 + 5z^2 + xz$ represents all odd positive integers [RO, Conjecture 1], and proved that the

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Generalized Riemann Hypothesis implies the truth of Conjecture 1.1 [RO, Theorem 7]. We prove

THEOREM 1.1. *Assuming the truth of Conjecture* 1.1, *the only ternary quadratic forms* $(a,b,c) := ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\ a \le b \le c)$ *that represent all positive integers* $n \equiv 4 \pmod 8$ *are the* 28 *forms*

$(1,1,2)$,	$(1,1,8)$,	$(1,1,10)$,	$(1,2,3)$,	$(1,2,4)$,	$(1,2,9)$,	$(1,2,11)$,
$(1,2,12)$,	$(1,2,16)$,	$(1,2,19)$,	$(1,3,8)$,	$(1,4,8)$,	$(1,5,6)$,	$(1,8,11)$,
$(1,8,12)$,	$(1,8,16)$,	$(1,8,19)$,	$(2,2,2)$,	$(2,2,8)$,	$(2,2,10)$,	$(2,3,4)$,
$(2,4,4)$,	$(2,4,12)$,	$(2,4,16)$,	$(3,4,8)$,	$(4,4,8)$,	$(4,8,12)$,	$(4,8,16)$.

Further we show that the forms in Theorem 1.1 can be identified by a finite set of integers that they represent.

THEOREM 1.2. *Let a, b and c denote positive integers. Under the assumption of the truth of Conjecture* 1.1, *if the ternary quadratic form* $ax^2 + by^2 + cz^2$ *represents the eight integers*

$$4, 12, 20, 28, 52, 60, 140, 308, \tag{1.3}$$

then it represents every positive integer $n \equiv 4 \pmod 8$.

The set of integers in (1.3) is minimal in the sense that for any one of the integers in the list (say h) there is a ternary quadratic form that does not represent h but does represent all the other integers in (1.3). Namely:

$(2,3,7)$ represents $12, 20, 28, 52, 60, 140, 308$ but not 4,
$(1,1,6)$ represents $4, 20, 28, 52, 60, 140, 308$ but not 12,
$(1,2,10)$ represents $4, 12, 28, 52, 60, 140, 308$ but not 20,
$(1,10,11)$ represents $4, 12, 20, 52, 60, 140, 308$ but not 28,
$(1,2,17)$ represents $4, 12, 20, 28, 60, 140, 308$ but not 52,
$(1,1,3)$ represents $4, 12, 20, 28, 52, 140, 308$ but not 60,
$(1,1,7)$ represents $4, 12, 20, 28, 52, 60, 308$ but not 140,
$(1,1,11)$ represents $4, 12, 20, 28, 52, 60, 140$ but not 308.

2. Proof of Theorems 1.1 and 1.2. We begin with some lemmas.

LEMMA 2.1. *The* 21 *ternary quadratic forms*

$(1,1,2)$,	$(1,1,8)$,	$(1,2,3)$,	$(1,2,4)$,	$(1,2,12)$,	$(1,2,16)$,	$(1,3,8)$,
$(1,4,8)$,	$(1,8,12)$,	$(1,8,16)$,	$(2,2,2)$,	$(2,2,8)$,	$(2,2,10)$,	$(2,3,4)$,
$(2,4,4)$,	$(2,4,12)$,	$(2,4,16)$,	$(3,4,8)$,	$(4,4,8)$,	$(4,8,12)$,	$(4,8,16)$,

represent all positive integers $n \equiv 4 \pmod 8$.

Proof. This is a consequence of (1.1) and (1.2), and the following two simple observations, where k, m and n denote positive integers:

(2.1) If n is represented by (a,b,c) then kn is represented by (ka,kb,kc).
(2.2) If n is represented by (a,b,c) and $m^2 \mid c$ then n is represented by $(a,b,c/m^2)$.

We just treat the form $(1,3,8)$ as the other forms can be handled similarly. Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. Then $n/4 \equiv 1 \pmod 2$. Hence, by (1.1), $n/4$ is represented by $(1,2,3)$. Thus, by (2.1), n is represented by $(4,8,12)$. Finally, by (2.2), as $2^2 \mid 4$ and $2^2 \mid 12$, n is represented by $(4/2^2, 8, 12/2^2) = (1,8,3)$ and so by $(1,3,8)$. ∎

LEMMA 2.2. *The ternary quadratic form* $(1,2,9)$ *represents all positive integers* $n \equiv 4 \pmod 8$.

Proof. Let $n \equiv 4 \pmod 8$. Then $n/2 \equiv 2 \pmod 4$. Hence, by (1.2), there are integers u, v and w such that

$$n/2 = u^2 + v^2 + w^2.$$

If $u \equiv v \pmod 3$ or $u \equiv w \pmod 3$ or $v \equiv w \pmod 3$, we may suppose that $u \equiv v \pmod 3$ by interchanging v and w or u and w as necessary. If $u \not\equiv v \pmod 3$, $u \not\equiv w \pmod 3$ and $v \not\equiv w \pmod 3$, we can permute u, v and w so that $(u,v,w) \equiv (2,1,0) \pmod 3$. Then replacing u by $-u$ we obtain $u \equiv v \pmod 3$. Hence we can define integers x, y and z by

$$x = u + v, \quad y = w, \quad z = (u - v)/3.$$

Then

$$x^2 + 2y^2 + 9z^2 = (u+v)^2 + 2w^2 + (u-v)^2 = 2(u^2 + v^2 + w^2) = n,$$

so that $(1,2,9)$ represents n. ∎

LEMMA 2.3. *The ternary form* $(1,1,10)$ *represents all positive integers* $n \equiv 4 \pmod 8$.

Proof. The ternary quadratic form $(1,1,10)$ is known as Ramanujan's form. Ramanujan [RA, p. 14] stated that "... *the even numbers which are not of the form* $x^2 + y^2 + 10z^2$ *are the numbers* $4^\lambda(16\mu + 6)$ ...". Dickson [D1, Corollary, p. 341] proved that $x^2+y^2+10z^2$ represents all even positive integers except those of the form $4^\lambda(16\mu+6)$, where λ and μ are nonnegative integers. As $4^\lambda(16\mu+6) \neq 8k+4$ the form $(1,1,10)$ represents every positive integer $n \equiv 4 \pmod 8$. ∎

LEMMA 2.4. *The ternary forms* $(1,2,11)$ *and* $(1,8,11)$ *represent all positive integers* $n \equiv 4 \pmod 8$.

Proof. Let $n \equiv 4 \pmod 8$. Then $n/4 \equiv 1 \pmod 2$. From Kaplansky [K, no. 12, p. 213] there are integers u, v and w such that

$$n/4 = u^2 + 2v^2 + 3w^2 + uw.$$

Define integers x, y and z by

$$x = 2u + w, \qquad y = v, \qquad z = w.$$

Then

$$x^2 + 8y^2 + 11z^2 = 4(u^2 + 2v^2 + 3w^2 + uw) = n,$$

so that $(1, 8, 11)$ represents n. Then, by (2.2), $(1, 2, 11)$ represents n. ∎

Lemma 2.5. *The ternary form* $(1, 5, 6)$ *represents all positive integers* $n \equiv 4 \pmod 8$.

Proof. Let $n \equiv 4 \pmod 8$. Then $n/4 \equiv 1 \pmod 2$. From Kaplansky [K, no. 14, p. 213] there are integers u, v and w such that

$$u^2 + 3v^2 + 3w^2 + uv + uw = n/4.$$

Define integers x, y and z by

$$x = 2u + v + w, \qquad y = v + w, \qquad z = v - w.$$

Then

$$x^2 + 5y^2 + 6z^2 = 4(u^2 + 3v^2 + 3w^2 + uv + uw) = n,$$

so that $(1, 5, 6)$ represents n. ∎

Lemma 2.6. *Assuming the truth of Conjecture* 1.1, *the ternary quadratic forms* $(1, 2, 19)$ *and* $(1, 8, 19)$ *represent all positive integers* $n \equiv 4 \pmod 8$.

Proof. Let $n \equiv 4 \pmod 8$. Then $n/4 \equiv 1 \pmod 2$. Assuming the truth of Conjecture 1.1, there are integers u, v and w such that

$$u^2 + 2v^2 + 5w^2 + uw = n/4.$$

Define integers x, y and z by

$$x = 2u + w, \qquad y = v, \qquad z = w.$$

Then

$$x^2 + 8y^2 + 19z^2 = 4(u^2 + 2v^2 + 5w^2 + uw) = n,$$

so that $(1, 8, 19)$ represents n. By (2.2), $(1, 2, 19)$ also represents n. ∎

Let n be an odd positive integer. Then $4n \equiv 4 \pmod 8$. If the ternary form $(1, 8, 19)$ represents all positive integers $\equiv 4 \pmod 8$ then there are integers x, y and z such that

$$x^2 + 8y^2 + 19z^2 = 4n.$$

Thus $x \equiv z \pmod 2$. Hence we can define integers u, v and w by

$$u = \frac{x - z}{2}, \qquad v = y, \qquad w = z,$$

and we obtain

$$n = \frac{x^2 + 8y^2 + 19z^2}{4} = u^2 + uw + 5w^2 + 2v^2.$$

Thus we have shown that the conjecture that the ternary quadratic form $(1,8,19)$ represents all positive integers $\equiv 4 \pmod 8$ is in fact equivalent to Conjecture 1.1.

Proof of Theorem 1.1. Suppose that the ternary form $ax^2+by^2+cz^2$, where a, b and c are positive integers with $a \le b \le c$, represents all positive integers $n \equiv 4 \pmod 8$.

As $ax^2+by^2+cz^2$ represents 4, we have

$$1 \le a \le 4.$$

If $a=1$, as $x^2+by^2+cz^2$ represents 12 we have

$$1 \le b \le 12.$$

If $a=b=1$, as $x^2+y^2 \ne 12$ and $x^2+y^2+cz^2$ represents 12 we have $1 \le c \le 12$, giving 12 candidate forms

$$x^2+y^2+cz^2, \quad c=1,2,\dots,12. \tag{2.1}$$

If $a=1$, $b=2$, as $x^2+2y^2 \ne 20$ and $x^2+2y^2+cz^2$ represents 20 we have $2 \le c \le 20$, giving 19 candidate forms

$$x^2+2y^2+cz^2, \quad c=2,3,\dots,20. \tag{2.2}$$

If $a=1$, $b=3$, as $x^2+3y^2 \ne 20$ and $x^2+3y^2+cz^2$ represents 20 we have $3 \le c \le 20$, giving 18 candidate forms

$$x^2+3y^2+cz^2, \quad c=3,4,\dots,20. \tag{2.3}$$

Continuing in this way, we arrive at $12+19+18+9+8+7+\cdots+2+9=236$ ternary forms to consider. They are

$$\begin{array}{llll}
(1,1,c), & c=1,\dots,12, & (2,3,c), & c=3,4,\\
(1,2,c), & c=2,\dots,20, & (2,4,c), & c=4,\dots,20,\\
(1,3,c), & c=3,\dots,20, & (3,3,c), & c=3,4,\\
(1,4,c), & c=4,\dots,12, & (3,4,c), & c=4,\dots,20,\\
(1,5,c), & c=5,\dots,12, & (4,4,c), & c=4,\dots,12,\\
(1,6,c), & c=6,\dots,12, & (4,5,c), & c=5,\dots,12,\\
(1,7,c), & c=7,\dots,12, & (4,6,c), & c=6,\dots,12,\\
(1,8,c), & c=8,\dots,20, & (4,7,c), & c=7,\dots,12,\\
(1,9,c), & c=9,\dots,12, & (4,8,c), & c=8,\dots,20,\\
(1,10,c), & c=10,11,12, & (4,9,c), & c=9,\dots,12,\\
(1,11,c), & c=11,\dots,28, & (4,10,c), & c=10,11,12,\\
(1,12,c), & c=12,\dots,20, & (4,11,c), & c=11,12,\\
(2,2,c), & c=2,\dots,12, & (4,12,c), & c=12,\dots,20.
\end{array}$$

It is easy to check that in this list of 236 forms, the 208 forms which are not in the list in Theorem 1.1 do not represent at least one of the eight integers $4, 12, 20, 28, 52, 60, 140, 308$ and so cannot represent all $n \equiv 4 \pmod 8$. The remaining 28 forms (those given in Theorem 1.1) do represent all $n \equiv 4 \pmod 8$ by Lemmas 2.1–2.6. ∎

Proof of Theorem 1.2. Suppose the ternary quadratic form $ax^2+by^2+cz^2$ ($a, b, c \in \mathbb{N}$) represents the integers $4, 12, 20, 28, 52, 60, 140, 308$. From the proof of Theorem 1.1 we have only to examine the 236 forms listed there. As already noted, 208 of these forms do not represent at least one of the eight integers $4, 12, 20, 28, 52, 60, 140, 308$. Thus (a, b, c) must be one of the remaining 28 forms listed in Theorem 1.1 and so represents all positive integers $n \equiv 4 \pmod 8$. ∎

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