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Analogues of Ramanujan's 24 squares formula

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Ramanujan's formula for the number $r_{24}(n)$ of representations of a positive integer n as a sum of 24 squares asserts that

$$r_{24}(n) = \frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}(n/2) + \frac{65536}{691}\sigma_{11}(n/4) + \frac{33152}{691}(-1)^{n-1}\tau(n) - \frac{65536}{691}\tau(n/2),$$

where

$$\sigma_{11}(k) = \begin{cases} \sum\limits_{\substack{d\in\mathbb{N}\\ d\mid k}} d^{11} & \text{if } k \in \mathbb{N}, \\ 0 & \text{if } k \in \mathbb{Q}\backslash\mathbb{N}, \end{cases}$$

and Ramanujan's function $\tau(n)$ is given by

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n, \quad q \in \mathbb{C}, \ |q| < 1.$$

In this paper, we determine a class of formulae analogous to Ramanujan's formula for $r_{24}(n)$.

Keywords: Ramanujan's 24 squares formula; sums of squares and triangular numbers; Ramanujan's tau function; sum of divisors function; Eisenstein series; infinite products; Dedekind eta function; modular forms.

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0. Notation

Let $\mathbb{N}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. Throughout this paper, q denotes a complex variable with $|q| < 1$. We now define the functions that we need.

For $k \in \mathbb{N}$ the infinite product E_k is defined by

$$E_k := \prod_{n \in \mathbb{N}} (1 - q^{kn}). \tag{0.1}$$

Ramanujan's theta functions φ and ψ are defined by

$$\varphi(q) := 1 + 2 \sum_{n \in \mathbb{N}} q^{n^2} = \sum_{n \in \mathbb{Z}} q^{n^2} \tag{0.2}$$

and

$$\psi(q) := \sum_{n \in \mathbb{N}_0} q^{n(n+1)/2}, \tag{0.3}$$

see [2, p. 6]. By Jacobi's triple product identity [2, p. 10] we have

$$\varphi(q) = E_1^{-2} E_2^5 E_4^{-2} \tag{0.4}$$

and

$$\psi(q) = E_1^{-1} E_2^2. \tag{0.5}$$

Ramanujan's tau function $\tau(n)$ $(n \in \mathbb{N})$ is given by

$$qE_1^{24} = \sum_{n \in \mathbb{N}} \tau(n) q^n, \tag{0.6}$$

see [15, Eq. (92), p. 151]. The first five values of τ are $\tau(1) = 1, \tau(2) = -24, \tau(3) = 252, \tau(4) = -1472$ and $\tau(5) = 4830$. The Bernoulli numbers B_n $(n \in \mathbb{N}_0)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad t \in \mathbb{C}, \ |t| < 2\pi, \tag{0.7}$$

so that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, $B_{14} = \frac{7}{6}, \ldots$ and $B_{2n+1} = 0$ $(n \in \mathbb{N})$. The classical normalized Eisenstein series $E_{2k}(q)$ $(k \in \mathbb{N})$ is defined by

$$E_{2k}(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n}. \tag{0.8}$$

The "sum of the powers of divisors" function $\sigma_k(n)$ is defined for $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ by

$$\sigma_k(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k. \tag{0.9}$$

For $m \notin \mathbb{N}$ we set $\sigma_k(m) := 0$. By taking $k = 6$ in (0.8), we obtain (as $B_{12} = -\frac{691}{2730}$)

$$E_{12}(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n. \tag{0.10}$$

Finally, if $f(q) = \sum_{n \in \mathbb{N}_0} f_n q^n$, we define $[f(q)]_n := f_n$, $n \in \mathbb{N}_0$.

1. Introduction

Ramanujan's famous formula [15, p. 162] for the number $r_{24}(n)$ of representations of a positive integer n as a sum of 24 squares asserts that

$$\begin{aligned} r_{24}(n) = {} & \frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}(n/2) + \frac{65536}{691}\sigma_{11}(n/4) \\ & + \frac{33152}{691}(-1)^{n-1}\tau(n) - \frac{65536}{691}\tau(n/2), \end{aligned} \tag{1.1}$$

where the sum of divisors function $\sigma_{11}(n)$ was defined in (0.9) and Ramanujan's tau function in (0.6). Ramanujan [13; 15, p. 153] conjectured and Mordell [10] proved that τ satisfies

$$\tau(kn) = \tau(k)\tau(n) - k^{11}\tau(n/k) \tag{1.2}$$

for all positive integers n and all primes k, see also [6, p. 297]. Taking $k = 2$ in (1.2) we have

$$\tau(2n) = -24\tau(n) - 2048\tau(n/2) \tag{1.3}$$

so that

$$(-1)^{n-1}\tau(n) = \tau(n) + 48\tau(n/2) + 4096\tau(n/4). \tag{1.4}$$

Using (1.4) in (1.1) we obtain Ramanujan's formula in the equivalent form

$$\begin{aligned} r_{24}(n) = {} & \frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}(n/2) + \frac{65536}{691}\sigma_{11}(n/4) \\ & + \frac{33152}{691}\tau(n) + \frac{1525760}{691}\tau(n/2) + \frac{135790592}{691}\tau(n/4). \end{aligned} \tag{1.5}$$

Multiplying both sides of (1.5) by q^n, and summing over $n \in \mathbb{N}_0$, we obtain (as $r_{24}(0) = 1$)

$$\begin{aligned} \sum_{n=0}^{\infty} r_{24}(n) q^n = {} & \frac{1}{4095}E_{12}(q) - \frac{2}{4095}E_{12}(q^2) + \frac{4096}{4095}E_{12}(q^4) \\ & + \frac{33152}{691}\Delta(q) + \frac{1525760}{691}\Delta(q^2) + \frac{135790592}{691}\Delta(q^4), \end{aligned} \tag{1.6}$$

where

$$\Delta(q) := \sum_{n=1}^{\infty} \tau(n) q^n (= qE_1^{24}). \tag{1.7}$$

Hence, as

$$\sum_{n=0}^{\infty} r_{24}(n)q^n = \varphi^{24}(q) = E_1^{-48}E_2^{120}E_4^{-48} \tag{1.8}$$

by (0.4), we have deduced the identity

$$\begin{aligned}\prod_{n=1}^{\infty}&(1-q^n)^{-48}(1-q^{2n})^{120}(1-q^{4n})^{-48}\\ &= \frac{1}{4095}E_{12}(q) - \frac{2}{4095}E_{12}(q^2) + \frac{4096}{4095}E_{12}(q^4)\\ &\quad + \frac{33152}{691}\Delta(q) + \frac{1525760}{691}\Delta(q^2) + \frac{135790592}{691}\Delta(q^4)\end{aligned} \tag{1.9}$$

from Ramanujan's 24 squares formula.

Conversely, using (1.8), (0.10) and (1.7) in (1.9), and then equating coefficients of q^n ($n \in \mathbb{N}$), we recover Ramanujan's 24 squares theorem in the form (1.5).

It is our aim in this paper to determine all identities like (1.9). In fact, for reasons that will be made clear in Sec. 2, we determine all identities of a slightly more general type than (1.9). We define

$$\Omega(q) := (\Delta(q)\Delta^2(q^4))^{1/3} = q^3\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{4n})^{16} = \sum_{n=1}^{\infty}\omega(n)q^n \tag{1.10}$$

(so that $\omega(1) = \omega(2) = 0$, $\omega(3) = 1$) and prove that there are precisely 28 identities of the type

$$\begin{aligned}q^r\prod_{n=1}^{\infty}&(1-q^n)^{a_1}(1-q^{2n})^{a_2}(1-q^{4n})^{a_4}\\ &= A_1E_{12}(q) + A_2E_{12}(q^2) + A_4E_{12}(q^4)\\ &\quad + B_1\Delta(q) + B_2\Delta(q^2) + B_4\Delta(q^4) + C\Omega(q),\end{aligned} \tag{1.11}$$

where r, a_1, a_2, a_4 are integers and $A_1, A_2, A_4, B_1, B_2, B_4, C$ are rational numbers. We prove in Sec. 2 the following result.

Theorem 1.1. *The only integers* r, a_1, a_2, a_4 *and rational numbers* $A_1, A_2, A_4, B_1, B_2, B_4, C$ *satisfying* (1.11) *are given in Table* 1.

We now begin our examination of the identities in Theorem 1.1. Each identity is identified by the number in the left-hand column of Table 1. Identity 1 is precisely formula (1.9) and so is equivalent to Ramanujan's 24 squares formula. Thus the identities of Theorem 1.1 are analogues of Ramanujan's 24 squares formula. Theorem 1.1 shows that there are exactly 10 formulae like (1.9), namely those with $C = 0$, that is, identities 1, 5, 6, 13, 14, 15, 16, 23, 24, 28.

Table 1. 28 identities of Theorem 1.1.

No.	a_1	a_2	a_4	r	A_1	A_2	A_4	B_1	B_2	B_4	C
1	−48	120	−48	0	$\frac{1}{4095}$	$\frac{-2}{4095}$	$\frac{4096}{4095}$	$\frac{33152}{691}$	$\frac{1525760}{691}$	$\frac{135790592}{691}$	0
2	−40	96	−32	1	$\frac{1}{65520}$	$\frac{-1}{65520}$	0	$\frac{690}{691}$	$\frac{42152}{691}$	8192	256
3	−32	72	−16	2	$\frac{1}{1048320}$	$\frac{-1}{1048320}$	0	$\frac{-1}{11056}$	$\frac{1123}{1382}$	256	16
4	−32	96	−40	0	0	$\frac{-1}{4095}$	$\frac{4096}{4095}$	32	$\frac{851328}{691}$	$\frac{45219840}{691}$	−4096
5	−24	48	0	3	$\frac{1}{16773120}$	$\frac{-1}{16773120}$	0	$\frac{-1}{176896}$	$\frac{-259}{22112}$	0	0
6	−24	72	−24	1	0	0	0	1	48	4096	0
7	−16	24	16	4	$\frac{1}{268369920}$	$\frac{-1}{268369920}$	0	$\frac{-1}{2830336}$	$\frac{-259}{353792}$	−1	$\frac{-1}{16}$
8	−16	48	−8	2	0	0	0	0	1	256	16
9	−16	72	−32	0	0	$\frac{-1}{4095}$	$\frac{4096}{4095}$	16	$\frac{320640}{691}$	$\frac{-65536}{694}$	−4096
10	−8	0	32	5	$\frac{1}{4293918720}$	$\frac{-1}{4293918720}$	0	$\frac{-1}{45285376}$	$\frac{-259}{5660672}$	$\frac{-1}{8}$	$\frac{-1}{256}$
11	−8	24	8	3	0	0	0	0	0	16	1
12	−8	48	−16	1	0	0	0	1	32	0	−256
13	0	−24	48	6	0	$\frac{1}{16773120}$	$\frac{-1}{16773120}$	0	$\frac{-1}{176896}$	$\frac{-259}{22112}$	0
14	0	0	24	4	0	0	0	0	0	1	0

(*Continued*)

Table 1. (*Continued*)

No.	a_1	a_2	a_4	r	A_1	A_2	A_4	B_1	B_2	B_4	C
15	0	24	0	2	0	0	0	0	1	0	0
16	0	48	−24	0	0	$\frac{-1}{4095}$	$\frac{4096}{4095}$	0	$\frac{-33152}{691}$	$\frac{-65536}{691}$	0
17	8	−24	40	5	$\frac{1}{4293918720}$	$\frac{-4097}{4293918720}$	$\frac{1}{1048320}$	$\frac{-1}{45285376}$	$\frac{253}{5660672}$	$\frac{345}{5528}$	$\frac{-1}{256}$
18	8	0	16	3	0	0	0	0	0	0	1
19	8	24	−8	1	0	0	0	1	16	0	−256
20	16	−24	32	4	$\frac{-1}{268369920}$	$\frac{4097}{268369920}$	$\frac{-1}{65520}$	$\frac{1}{2830336}$	$\frac{-253}{353792}$	$\frac{1}{691}$	$\frac{1}{16}$
21	16	0	8	2	0	0	0	0	1	0	−16
22	16	24	−16	0	0	$\frac{-1}{4095}$	$\frac{4096}{4095}$	−16	$\frac{-210048}{691}$	$\frac{-65536}{691}$	4096
23	24	−24	24	3	$\frac{1}{16773120}$	$\frac{-4097}{16773120}$	$\frac{1}{4095}$	$\frac{-1}{176896}$	$\frac{253}{22112}$	$\frac{-16}{691}$	0
24	24	0	0	1	0	0	0	1	0	0	0
25	32	−24	16	2	$\frac{-1}{1048320}$	$\frac{4097}{1048320}$	$\frac{-16}{4095}$	$\frac{1}{11056}$	$\frac{1129}{1382}$	$\frac{256}{691}$	−16
26	32	0	−8	0	0	$\frac{-1}{4095}$	$\frac{4096}{4095}$	−32	$\frac{-210048}{691}$	$\frac{-65536}{691}$	4096
27	40	−24	8	1	$\frac{1}{65520}$	$\frac{-4097}{65520}$	$\frac{256}{4095}$	$\frac{690}{691}$	$\frac{-9032}{691}$	$\frac{-4096}{691}$	256
28	48	−24	0	0	$\frac{1}{4095}$	$\frac{4096}{4095}$	0	$\frac{-33152}{691}$	$\frac{-66536}{691}$	0	0

Since

$$\prod_{n=1}^{\infty}(1-(-q)^n)=\prod_{n=1}^{\infty}(1-q^n)^{-1}(1-q^{2n})^3(1-q^{4n})^{-1}, \tag{1.12}$$

we see that the mapping $q \mapsto -q$ transforms the left-hand side of (1.11) as follows:

$$q^r\prod_{n=1}^{\infty}(1-q^n)^{a_1}(1-q^{2n})^{a_2}(1-q^{4n})^{a_4}$$
$$\mapsto (-1)^r q^r \prod_{n=1}^{\infty}(1-q^n)^{-a_1}(1-q^{2n})^{3a_1+a_2}(1-q^{4n})^{-a_1+a_4}. \tag{1.13}$$

To determine the effect of $q \mapsto -q$ on the right-hand side of (1.11), we must determine $E_{12}(-q), \Delta(-q)$ and $\Omega(-q)$.

Proposition 1.1.

(i) $E_{12}(-q)=-E_{12}(q)+4098E_{12}(q^2)-4096E_{12}(q^4)$,
(ii) $\Delta(-q)=-\Delta(q)-48\Delta(q^2)-4096\Delta(q^4)$,
(iii) $\Omega(-q)=-\Omega(q)-16\Delta(q^4)$.

Proof. (i) By (0.10) we have

$$E_{12}(q)+E_{12}(-q)=2+\frac{2\cdot 65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(2n)q^{2n}. \tag{1.14}$$

The identity

$$\sigma_{11}(2n)=2049\sigma_{11}(n)-2048\sigma_{11}(n/2), \quad n\in\mathbb{N}, \tag{1.15}$$

is easily proved. Using (1.15) in (1.14), and appealing to (0.10), we obtain

$$\sum_{n=1}^{\infty}\sigma_{11}(2n)q^{2n}=\frac{2049\cdot 691}{65520}E_{12}(q^2)-\frac{2048\cdot 691}{65520}E_{12}(q^4)-\frac{691}{65520}. \tag{1.16}$$

Putting (1.16) into (1.14), we deduce

$$E_{12}(q)+E_{12}(-q)=4098E_{12}(q^2)-4096E_{12}(q^4)$$

from which (i) follows.

(ii) From (1.7) and (1.4) we obtain

$$\Delta(-q)=\sum_{n=1}^{\infty}(-1)^n\tau(n)q^n=-\sum_{n=1}^{\infty}(\tau(n)+48\tau(n/2)+4096\tau(n/4))q^n$$

from which (ii) follows.

(iii) By [1, Eq. (3.1), p. 242] we have

$$E_2^{24}=E_1^{16}E_4^8+16qE_1^8E_4^{16}. \tag{1.17}$$

Multiplying (1.17) by $-q^3E_1^{-8}E_4^8$, we obtain the identity

$$-q^3E_1^{-8}E_2^{24}E_4^8 = -q^3E_1^8E_4^{16} - 16q^4E_4^{24}. \tag{1.18}$$

By (1.10) and (1.7) the right-hand side of (1.18) is $-\Omega(q) - 16\Delta(q^4)$. By (1.12) and (1.10) the left-hand side of (1.18) is $-q^3(E_1^{-1}E_2^3E_4^{-1})^8E_4^{16} = \Omega(-q)$. This completes the proof of (iii). □

From (1.13) and Proposition 1.1 we see that the identity (1.11) with parameters

$$r, a_1, a_2, a_4, A_1, A_2, A_4, B_1, B_2, B_4, C$$

transforms under $q \mapsto -q$ into the identity (1.11) with parameters

$$r', a_1', a_2', a_4', A_1', A_2', A_4', B_1', B_2', B_4', C',$$

where

$$\begin{cases} r' = r, a_1' = -a_1, a_2' = 3a_1 + a_2, a_4' = -a_1 + a_4, \\ A_1' = (-1)^{r+1}A_1, A_2' = (-1)^r(-4098A_1 + A_2), \\ A_4' = (-1)^r(-4096A_1 + A_4), B_1' = (-1)^{r+1}B_1, \\ B_2' = (-1)^r(-48B_1 + B_2), \\ B_4' = (-1)^r(-4096B_1 + B_4 - 16C), C' = (-1)^{r+1}C. \end{cases}$$

Thus the identities $17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28$ in Theorem 1.1 are obtained from the identities $10, 11, 12, 7, 8, 9, 5, 6, 3, 4, 2, 1$ respectively by changing q to $-q$, and thus are not really new. Hence, we need only to consider the identities 1–16 in Theorem 1.1.

First we observe that the identities $6, 11, 14$ and 15 are trivial. Identity 6 is

$$qE_1^{-24}E_2^{72}E_4^{-24} = \Delta(q) + 48\Delta(q^2) + 4096\Delta(q^4).$$

This follows by mapping $q \mapsto -q$ in $\Delta(q) = qE_1^{24}$ and appealing to (1.12) and Proposition 1.1(ii). Identity 11 is

$$q^3E_1^{-8}E_2^{24}E_4^8 = 16\Delta(q^4) + \Omega(q).$$

This follows by mapping $q \mapsto -q$ in $\Omega(q) = q^3E_1^8E_4^{16}$ and appealing to (1.12) and Proposition 1.1(iii). Identities 14 and 15 are

$$q^4E_4^{24} = \Delta(q^4), \quad q^2E_2^{24} = \Delta(q^2),$$

respectively, which follow from $\Delta(q) = qE_1^{24}$ by mapping $q \mapsto q^4$ and $q \mapsto q^2$ respectively.

Further, identities 13 and 16 result from mapping $q \mapsto q^2$ in identities 5 and 28 respectively.

Thus we are left to consider the identities 1–5, 7–10 and 12. Of these 10 identities we find that two are known (identities 1 and 5) and the remaining eight are new (see Theorems 1.2–1.9).

By considering the first few terms in the expansion of $E_1^{b_1}E_2^{b_2}E_4^{b_4}E_8^{b_8}E_{16}^{b_{16}}$ ($b_1, b_2, b_4, b_8, b_{16} \in \mathbb{Z}$) in powers of q, it is easy to check that

$$E_1^{b_1}E_2^{b_2}E_4^{b_4}E_8^{b_8}E_{16}^{b_{16}} = 1 \quad \text{if and only if } b_1 = b_2 = b_4 = b_8 = b_{16} = 0.$$

Using this result, together with (0.4) and (0.5), it is easy to prove that

$$E_1^{a_1}E_2^{a_2}E_4^{a_4} = \varphi^t(q)\varphi^u(q^2)\varphi^v(q^4)\psi^w(q)\psi^x(q^2)\psi^y(q^4)\psi^z(q^8)$$

for some integers t, u, v, w, x, y, z if and only if

$$4a_1 + 2a_2 + a_4 = 0. \tag{1.19}$$

The condition (1.19) is satisfied by identities 1, 2, 3, 5, 7 and 10 but not by identities 4, 8, 9 and 12.

If (1.19) is satisfied, then

$$E_1^{a_1}E_2^{a_2}E_4^{a_4} = \psi^{-a_1}(q)\psi^{-2a_1-a_2}(q^2). \tag{1.20}$$

Thus, as $\varphi(q)\psi^{-2}(q)\psi(q^2) = 1$, we have

$$E_1^{a_1}E_2^{a_2}E_4^{a_4} = \varphi^t(q)\psi^{-2t-a_1}(q)\psi^{t-2a_1-a_2}(q^2) \tag{1.21}$$

for any integer t.

Identity 1. As we have already mentioned this identity is formula (1.9). Equating the coefficients of q^n $(n \in \mathbb{N})$, we recover Ramanujan's 24 squares formula (1.5).

Identity 2. Taking $a_1 = -40, a_2 = 96, a_4 = -32$ and $t = 16$ in (1.21), identity 2 becomes

$$\begin{aligned} q\varphi^{16}(q)\psi^8(q) &= qE_1^{-40}E_2^{96}E_4^{-32} \\ &= \frac{1}{65520}E_{12}(q) - \frac{1}{65520}E_{12}(q^2) + \frac{690}{691}\Delta(q) \\ &\quad + \frac{42152}{691}\Delta(q^2) + 8192\Delta(q^4) + 256\Omega(q). \end{aligned} \tag{1.22}$$

Equating the coefficients of q^n $(n \in \mathbb{N})$ in (1.22), we obtain the following new result analogous to Ramanujan's 24 squares formula.

Theorem 1.2. *Let $n \in \mathbb{N}$. Then the number of representations of $n-1$ as a sum of 16 squares and 8 triangular numbers is*

$$\begin{aligned} r(n-1 = 16\square + 8\triangle) &= \frac{1}{691}\sigma_{11}(n) - \frac{1}{691}\sigma_{11}(n/2) + \frac{690}{691}\tau(n) \\ &\quad + \frac{42152}{691}\tau(n/2) + 8192\tau(n/4) + 256\omega(n). \end{aligned}$$

Identity 3. Taking $a_1 = -32, a_2 = 72, a_4 = -16$ and $t = 8$ in (1.21), identity 3 becomes

$$\begin{aligned} q^2\varphi^8(q)\psi^{16}(q) &= q^2E_1^{-32}E_2^{72}E_4^{-16} \\ &= \frac{1}{1048320}E_{12}(q) - \frac{1}{1048320}E_{12}(q^2) - \frac{1}{11056}\Delta(q) \\ &\quad + \frac{1123}{1382}\Delta(q^2) + 256\Delta(q^4) + 16\Omega(q). \end{aligned} \tag{1.23}$$

Equating the coefficients of $q^n (n \in \mathbb{N}, n \geq 2)$ in (1.23), we obtain another new formula analogous to Ramanujan's 24 squares formula.

Theorem 1.3. *Let $n \in \mathbb{N}$ satisfy $n \geq 2$. Then the number of representations of $n-2$ as a sum of* 8 *squares and* 16 *triangular numbers is*

$$\begin{aligned} r(n-2 = 8\square + 16\triangle) &= \frac{1}{11056}\sigma_{11}(n) - \frac{1}{11056}\sigma_{11}(n/2) - \frac{1}{11056}\tau(n) \\ &\quad + \frac{1123}{1382}\tau(n/2) + 256\tau(n/4) + 16\omega(n). \end{aligned}$$

Identity 4. Equating the coefficients of q^n $(n \in \mathbb{N})$ in identity 4, we obtain the following new result.

Theorem 1.4. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} [E_1^{-32}E_2^{96}E_4^{-40}]_n &= -\frac{16}{691}\sigma_{11}(n/2) + \frac{65536}{691}\sigma_{11}(n/4) + 32\tau(n) \\ &\quad + \frac{851328}{691}\tau(n/2) + \frac{45219840}{691}\tau(n/4) - 4096\omega(n). \end{aligned}$$

Identity 5. Taking $a_1 = -24$, $a_2 = 48$, $a_4 = 0$ and $t = 0$ in (1.21), identity 5 becomes

$$\begin{aligned} q^3\psi^{24}(q) = q^3E_1^{-24}E_2^{48} &= \frac{1}{16773120}E_{12}(q) - \frac{1}{16773120}E_{12}(q^2) \\ &\quad - \frac{1}{176896}\Delta(q) - \frac{259}{22112}\Delta(q^2). \end{aligned}$$

Equating the coefficients of $q^n (n \in \mathbb{N}, n \geq 3)$, we deduce

$$t_{24}(n-3) = \frac{1}{176896}\sigma_{11}(n) - \frac{1}{176896}\sigma_{11}(n/2) - \frac{1}{176896}\tau(n) - \frac{259}{22112}\tau(n/2),$$

where $t_{24}(n-3)$ denotes the number of representations of the nonnegative integer $n-3$ as a sum of 24 triangular numbers. A general formula for the number $t_{2s}(n)$ of representations of n as a sum of $2s$ triangular numbers was first given by Ramanujan [14, Sec. 12; 15, pp. 190–191]. Cooper [4, Theorems 3.5 and 3.6] has given an elementary proof of Ramanujan's formula. The special case $2s = 24$

of Ramanujan's formula gives the above formula for $t_{24}(n-3)$. The formula for $t_{24}(n-3)$ has been stated explicitly by Ono, Robins and Wahl [12, Theorem 8, p. 86] and Cooper [4, p. 137].

Identity 7. Taking $a_1 = -16, a_2 = 24, a_4 = 16$ and $t = 0$ in (1.21), identity 7 becomes

$$\begin{aligned} q^4\psi^{16}(q)\psi^8(q^2) &= q^4 E_1^{-16}E_2^{24}E_4^{16} \\ &= \frac{1}{268369920}E_{12}(q) - \frac{1}{268369920}E_{12}(q^2) \\ &\quad - \frac{1}{2830336}\Delta(q) - \frac{259}{353792}\Delta(q^2) - \Delta(q^4) - \frac{1}{16}\Omega(q). \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N}, n \geq 4)$, we obtain the following result.

Theorem 1.5. *Let $n \in \mathbb{N}$ satisfy $n \geq 4$. Then the number of representations of $n-4$ as a sum of 16 triangular numbers plus twice the sum of 8 triangular numbers is*

$$\begin{aligned} r(n-4 = 16\triangle + 2(8\triangle)) &= \frac{1}{2830336}\sigma_{11}(n) - \frac{1}{2830336}\sigma_{11}(n/2) - \frac{1}{2830336}\tau(n) \\ &\quad - \frac{259}{353792}\tau(n/2) - \tau(n/4) - \frac{1}{16}\omega(n). \end{aligned}$$

Identity 8. Equating the coefficients of q^n $(n \in \mathbb{N}, n \geq 2)$ in identity 8 gives the following result.

Theorem 1.6. *Let $n \in \mathbb{N}$ satisfy $n \geq 2$. Then*

$$[q^2 E_1^{-16}E_2^{48}E_4^{-8}]_n = \tau(n/2) + 256\tau(n/4) + 16\omega(n).$$

Identity 9. Equating the coefficients of q^n $(n \in \mathbb{N})$ in identity 9, we obtain the following result.

Theorem 1.7. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} [E_1^{-16}E_2^{72}E_4^{-32}]_n &= -\frac{16}{691}\sigma_{11}(n/2) + \frac{65536}{691}\sigma_{11}(n/4) + 16\tau(n) \\ &\quad + \frac{320640}{691}\tau(n/2) - \frac{65536}{691}\tau(n/4) - 4096\omega(n). \end{aligned}$$

Identity 10. Taking $a_1 = -8, a_2 = 0, a_4 = 32$ and $t = 0$ in (1.21), identity 10 becomes

$$\begin{aligned} q^5\psi^8(q)\psi^{16}(q^2) &= q^5 E_1^{-8}E_4^{32} \\ &= \frac{1}{4293918720}E_{12}(q) - \frac{1}{4293918720}E_{12}(q^2) \\ &\quad - \frac{1}{45285376}\Delta(q) - \frac{259}{5660672}\Delta(q^2) - \frac{1}{8}\Delta(q^4) - \frac{1}{256}\Omega(q). \end{aligned}$$

Equating the coefficients of q^n $(n \in \mathbb{N}, n \geq 5)$, we obtain the following result.

Theorem 1.8. *Let $n \in \mathbb{N}$ satisfy $n \geq 5$. Then the number of representations of $n-5$ as a sum of 8 triangular numbers plus twice the sum of 16 triangular numbers is*

$$\begin{aligned} r(n-5 = 8\triangle + 2(16\triangle)) = {} & \frac{1}{45285376}\sigma_{11}(n) - \frac{1}{45285376}\sigma_{11}(n/2) \\ & - \frac{1}{45285376}\tau(n) - \frac{259}{5660672}\tau(n/2) - \frac{1}{8}\tau(n/4) \\ & - \frac{1}{256}\omega(n). \end{aligned}$$

Identity 12. Equating the coefficients of q^n $(n \in \mathbb{N})$ in identity 12 gives the following result.

Theorem 1.9. *Let $n \in \mathbb{N}$. Then*

$$[qE_1^{-8}E_2^{48}E_4^{-16}]_n = \tau(n) + 32\tau(n/2) - 256\omega(n).$$

2. Proof of Theorem 1.1.

Let $\mathcal{H}$ denote the Poincaré upper half-plane, that is $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. For $z \in \mathcal{H}$ the Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) := e^{2\pi i z/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}),$$

see [5, p. 19] for example. Let Γ denote the modular group, that is

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The Dedekind eta function $\eta(z)$ is a holomorphic modular form of weight 1/2 for Γ with a certain multiplier system, which we denote by ν_η. For all $M \in \Gamma, \nu_\eta(M)$ is a 24th root of unity depending only on M. The determination of $\nu_\eta(M)$ was first addressed by Dedekind in the nineteenth century, then by Rademacher in 1931 and later by Petersson in 1954. As we need the value of $\nu_\eta(M)$ explicitly, we now state Petersson's formula in the form given by Knopp [7, p. 51]. In order to do this we introduce the notation $\left(\frac{c}{d}\right)_*$ and $\left(\frac{d}{c}\right)^*$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, so that $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. If $c = 0$, then $d = \pm 1$ and we define

$$\left(\frac{0}{1}\right)_* := 1, \quad \left(\frac{0}{-1}\right)_* := -1.$$

If c is even and $c \neq 0$, then d is odd and we define

$$\left(\frac{c}{d}\right)_* := \left(\frac{c}{|d|}\right) (-1)^{(\mathrm{sgn}(c)-1)(\mathrm{sgn}(d)-1)/4},$$

where $(\frac{c}{|d|})$ is the Jacobi symbol of quadratic reciprocity and $\mathrm{sgn}(x) = \frac{x}{|x|}$ for any nonzero real number x. If c is odd and $d = 0$, then $c = \pm 1$ and we define

$$\left(\frac{0}{1}\right)^* := 1, \quad \left(\frac{0}{-1}\right)^* := 1.$$

Finally, if c is odd and $d \neq 0$, we define

$$\left(\frac{d}{c}\right)^* := \left(\frac{d}{|c|}\right).$$

We can now state Petersson's formula.

Theorem 2.1. *For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ the multiplier system of the Dedekind eta function $\eta(z)$ is given by*

$$\eta_\nu(M) = \left(\frac{d}{c}\right)^* e^{2\pi i((a+d)c-bd(c^2-1)-3c)/24} \quad \text{if } c \text{ is odd}$$

and

$$\eta_\nu(M) = \left(\frac{c}{d}\right)_* e^{2\pi i((a+d)c-bd(c^2-1)+3d-3-3cd)/24} \quad \text{if } c \text{ is even.}$$

For a positive integer N, we define the Hecke congruence group of level N by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \bmod N \right\}.$$

Clearly $\Gamma_0(1) = \Gamma$, $\Gamma_0(N) \leq \Gamma$ and $\Gamma_0(kN) \leq \Gamma_0(N)$ for any positive integer k. We denote the complex vector space of holomorphic modular forms of weight k for $\Gamma_0(N)$ with trivial multiplier system by $M_k(\Gamma_0(N))$.

We now address the question: Under what conditions does the eta quotient $\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z)$ $(a_1, a_2, a_4 \in \mathbb{Z})$ belong to $M_{12}(\Gamma_0(4))$? This kind of question was probably first addressed by Newman [11].

Theorem 2.2. *Let $a_1, a_2, a_4 \in \mathbb{Z}$. Then*

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z) \in M_{12}(\Gamma_0(4))$$

if and only if

$$a_1 + a_2 + a_4 = 24, \tag{2.1}$$

$$a_1 + 2a_2 + 4a_4 \geq 0, \tag{2.2}$$

$$a_1 + 2a_2 + a_4 \geq 0, \tag{2.3}$$

$$4a_1 + 2a_2 + a_4 \geq 0, \tag{2.4}$$

and

$$8|a_1, \quad 24|a_2, \quad 8|a_4. \tag{2.5}$$

Proof. Suppose first that $a_1, a_2, a_4 \in \mathbb{Z}$ satisfy (2.1)–(2.5). We show that $g(z) := \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z)$ is a holomorphic modular form of weight 12 for $\Gamma_0(4)$ with trivial multiplier system.

From the infinite product representation of $\eta(z)$ we see that $\eta(z)$ is a nonzero holomorphic function on $\mathcal{H}$. Thus $g(z)$ is a holomorphic function on $\mathcal{H}$ (for any $a_1, a_2, a_4 \in \mathbb{Z}$). By (2.2)–(2.4) we have

$$\sum_{\substack{m \in \mathbb{N} \\ m \mid 4}}^{\infty} \frac{(c,m)^2}{m} a_m \geq 0$$

for all $c \in \mathbb{N}$ with $c \mid 4$. Thus by [9, Corollary 2.3, p. 37], $g(z)$ is holomorphic at all cusps (including ∞). Since $\eta(z)$ is a modular form of weight 1/2 for Γ with multiplier system ν_η, $g(z)$ transforms like a modular form of weight

$$\frac{1}{2}(a_1 + a_2 + a_4) = 12$$

for $\Gamma_0(4)$ with multiplier system ν_g given by

$$\nu_g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{a_1} \nu_\eta\begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}^{a_2} \nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}^{a_4},$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. Thus to complete the proof that $g(z)$ belongs to $M_{12}(\Gamma_0(4))$ we have only to prove that

$$\nu_g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4).$$

Now by (2.1) we have

$$\begin{aligned} \nu_g\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{a_1} \nu_\eta\begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}^{a_2} \nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}^{24-a_1-a_2} \\ &= \left(\frac{\nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}}\right)^{a_1} \end{aligned}$$

as ν_η is a 24th root of unity and by (2.5) we have $24 \mid a_2$. Since $8 \mid a_1$ (by (2.5)) we complete the proof by showing that

$$\left(\frac{\nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}}\right)^{8} = 1 \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4). \tag{2.6}$$

Suppose first that $c \equiv 0 \pmod 8$ so that $c/4$ is even. Then, by Petersson's theorem (Theorem 2.1), we have with $\omega = e^{2\pi i/24}$

$$\frac{\nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}} = \frac{\left(\frac{c}{d}\right)_* \omega^{(a+d)c-bd(c^2-1)+3d-3-3cd}}{\left(\frac{c/4}{d}\right)_* \omega^{(a+d)(c/4)-4bd((c/4)^2-1)+3d-3-3(c/4)d}}$$

$$= \pm\omega^{\frac{3(a+d)c}{4}-\frac{3bdc^2}{4}-3bd-\frac{9cd}{4}}$$

so that

$$\left(\frac{\nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}}\right)^8 = \omega^{6(a+d)c-6bdc^2-24bd-18cd}.$$

Now $c \equiv 0 \pmod 8$ implies

$$6(a+d)c - 6bdc^2 - 24bd - 18cd \equiv 0 \pmod{24}$$

so that (2.6) holds.

Now suppose that $c \equiv 4 \pmod 8$ so that $c/4$ is odd. Then, by Theorem 2.1, we have

$$\frac{\nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}} = \frac{\left(\frac{c}{d}\right)_* \omega^{(a+d)c-bd(c^2-1)+3d-3-3cd}}{\left(\frac{d}{c/4}\right)^* \omega^{(a+d)(c/4)-4bd((c/4)^2-1)-3c/4}}$$

$$= \pm\omega^{\frac{3(a+d)c}{4}-\frac{3bdc^2}{4}-3bd+3d-3-3cd+\frac{3c}{4}}$$

so that

$$\left(\frac{\nu_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\nu_\eta\begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}}\right)^8 = \omega^{6(a+d)c-6bdc^2-24bd+24d-24-24cd+6c}.$$

Now $c \equiv 0 \pmod 4$ implies

$$6(a+d)c - 6bdc^2 - 24bd + 24d - 24 - 24cd + 6c \equiv 0 \pmod{24}$$

so that (2.6) holds. This completes the proof that $g(z) \in M_{12}(\Gamma_0(4))$.

Conversely, suppose that $a_1, a_2, a_4 \in \mathbb{Z}$ are such that

$$g(z) := \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z) \in M_{12}(\Gamma_0(4)).$$

We prove that (2.1)–(2.5) hold. As $g(z)$ is a modular form of weight 12, Eq. (2.1) holds. As $g(z)$ is holomorphic at all cusps, inequalities (2.2)–(2.4) hold by [9, Corollary 2.3, p. 37]. Thus we are left to prove (2.5). As g has a trivial multiplier system for $\Gamma_0(4)$ we have

$$\nu_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{a_1} \nu_\eta \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}^{a_2} \nu_\eta \begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix}^{24-a_1-a_2} = 1$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. Choose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 8 & 9 \end{smallmatrix}\right) \in \Gamma_0(4)$. Then by Theorem 2.1, we have with $\omega = e^{2\pi\imath/24}$

$$\nu_\eta \begin{pmatrix} 1 & 1 \\ 8 & 9 \end{pmatrix} = \omega^{17}, \quad \nu_\eta \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} = \omega^{22}, \quad \nu_\eta \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix} = \omega^{2},$$

so that

$$17a_1 + 22a_2 + 2(24 - a_1 - a_2) \equiv 0 \pmod{24},$$

and thus

$$3a_1 + 4a_2 \equiv 0 \pmod{24}. \tag{2.7}$$

Next choose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) \in \Gamma_0(4)$. Here

$$\nu_\eta \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \omega^{20}, \quad \nu_\eta \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \omega^{22}, \quad \nu_\eta \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \omega^{23},$$

so that

$$20a_1 + 22a_2 + 23(24 - a_1 - a_2) \equiv 0 \pmod{24},$$

and hence

$$3a_1 + a_2 \equiv 0 \pmod{24}. \tag{2.8}$$

From (2.8) we deduce $a_2 \equiv 0 \pmod 3$. Subtracting (2.8) from (2.7), we obtain $3a_2 \equiv 0 \pmod{24}$ so $a_2 \equiv 0 \pmod 8$. Hence $a_2 \equiv 0 \pmod{24}$ and (by (2.8)) $a_1 \equiv 0 \pmod 8$. Finally $a_4 = 24 - a_1 - a_2 \equiv 0 \pmod 8$. This completes the proof of (2.5).

The proof of Theorem 2.2 is now complete. □

We now determine the integers a_1, a_2, a_4 satisfying (2.1)–(2.5).

Theorem 2.3. *The only integers a_1, a_2, a_4 satisfying* (2.1)–(2.5) *are given in Table* 2.

Proof. Let $a_1, a_2, a_4 \in \mathbb{Z}$ satisfy (2.1)–(2.5). By (2.5) we have $a_1 = 8A_1$ and $a_2 = 24A_2$ for some integers A_1 and A_2. Then (2.1)–(2.4) give

$$a_4 = 24 - 8A_1 - 24A_2, \tag{2.9}$$

$$A_1 + 2A_2 \leq 4, \tag{2.10}$$

$$A_2 \geq -1, \tag{2.11}$$

$$A_1 + A_2 \geq -1. \tag{2.12}$$

Table 2. Values of a_1, a_2, a_4 satisfying (2.1)–(2.5).

No.	a_1	a_2	a_4	no	a_1	a_2	a_4
1	−48	120	−48	15	0	24	0
2	−40	96	−32	16	0	48	−24
3	−32	72	−16	17	8	−24	40
4	−32	96	−40	18	8	0	16
5	−24	48	0	19	8	24	−8
6	−24	72	−24	20	16	−24	32
7	−16	24	16	21	16	0	8
8	−16	48	−8	22	16	24	−16
9	−16	72	−32	23	24	−24	24
10	−8	0	32	24	24	0	0
11	−8	24	8	25	32	−24	16
12	−8	48	−16	26	32	0	−8
13	0	−24	48	27	40	−24	8
14	0	0	24	28	48	−24	0

Thus $-6 \leq A_1 \leq 6$ and $-1 \leq A_2 \leq 5$. A simple computer search through the lattice points of the box $\{(x, y) \in \mathbb{Z}^2 \mid -6 \leq x \leq 6, -1 \leq y \leq 5\}$ yielded 28 lattice points (A_1, A_2) satisfying (2.9)–(2.12). The corresponding 28 values of (a_1, a_2, a_4) are listed in Table 2. $\square$

For $z \in \mathcal{H}$ and $k \in \mathbb{N}$ we define the Eisenstein series $\xi_{2k}(z)$ by

$$\xi_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n z}$$

so that with $q = e^{2\pi i z}$ we have $E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n = \xi_{2k}(z)$. It is well known that $\xi_{2k}(z)$ is a modular form of weight $2k$ for the full modular group Γ with trivial multiplier system if $k \geq 2$ [9, p. 19]. Thus in particular we have

$$\xi_{12}(z), \quad \xi_{12}(2z), \quad \xi_{12}(4z) \in M_{12}(\Gamma_0(4)).$$

Theorem 2.4. *A basis for the complex vector space* $M_{12}(\Gamma_0(4))$ *comprises*

$$\xi_{12}(z), \quad \xi_{12}(2z), \quad \xi_{12}(4z), \quad \eta^{24}(z), \quad \eta^{24}(2z), \quad \eta^{24}(4z), \quad \eta^{8}(z)\eta^{16}(4z).$$

Proof. By the corollary to Proposition 4 in [8] we have

$$\dim(M_{12}(\Gamma_0(4))) = 7.$$

We have already noted that

$$\xi_{12}(z), \quad \xi_{12}(2z), \quad \xi_{12}(4z) \in M_{12}(\Gamma_0(4)), \tag{2.13}$$

and, by Theorem 2.2, we have

$$\eta^{24}(z), \quad \eta^{24}(2z), \quad \eta^{24}(4z), \quad \eta^{8}(z)\eta^{16}(4z) \in M_{12}(\Gamma_0(4)). \tag{2.14}$$

The seven modular forms listed in (2.13) and (2.14) are linearly independent over $\mathbb{C}$. Hence they form a basis for $M_{12}(\Gamma_0(4))$. □

By [9, Corollary 2.3, p. 37] $\eta^{24}(z), \eta^{24}(2z), \eta^{24}(4z)$ and $\eta^8(z)\eta^{16}(4z)$ are cuspidal eta products.

Proof of Theorem 1.1. Let r, a_1, a_2, a_4 be integers and $A_1, A_2, A_4, B_1, B_2, B_4, C$ be rational numbers such that (1.11) holds. Set $q = e^{2\pi i z}$. As $|q| < 1$ we have $z \in \mathcal{H}$. Now

$$\begin{aligned} E_{12}(q^k) &= E_{12}(e^{2\pi i k z}) = \xi_{12}(kz), \quad k = 1, 2, 4, \\ \Delta(q^k) &= \eta^{24}(kz), \quad k = 1, 2, 4, \\ \Omega(q) &= \eta^8(z)\eta^{16}(4z), \end{aligned}$$

and

$$q^r \prod_{n=1}^{\infty} (1 - q^n)^{a_1}(1 - q^{2n})^{a_2}(1 - q^{4n})^{a_4} = q^{r - \frac{a_1}{24} - \frac{a_2}{12} - \frac{a_4}{6}} \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z),$$

so that (1.11) becomes

$$\begin{aligned} &q^{r - \frac{a_1}{24} - \frac{a_2}{12} - \frac{a_4}{6}} \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z) \\ &= A_1\xi_{12}(z) + A_2\xi_{12}(2z) + A_4\xi_{12}(4z) \\ &\quad + B_1\eta^{24}(z) + B_2\eta^{24}(2z) + B_4\eta^{24}(4z) + C\eta^8(z)\eta^{16}(4z). \end{aligned}$$

The right-hand side belongs to $M_{12}(\Gamma_0(4))$. Hence

$$q^{r - \frac{a_1}{24} - \frac{a_2}{12} - \frac{a_4}{6}} \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z) \in M_{12}(\Gamma_0(4)).$$

Since $\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z)$ is a modular form of weight $\frac{1}{2}(a_1 + a_2 + a_4)$ with respect to some multiplier system, it must be the case that $q^{r - \frac{a_1}{24} - \frac{a_2}{12} - \frac{a_4}{6}}$ transforms like a modular form with respect to some multiplier system. However this is clearly not possible if $r - \frac{a_1}{24} - \frac{a_2}{12} - \frac{a_4}{6} \neq 0$. Hence we must have $r = \frac{a_1}{24} + \frac{a_2}{12} + \frac{a_4}{6}$ and thus

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z) \in M_{12}(\Gamma_0(4)).$$

Then, by Theorem 2.2, we see that a_1, a_2, a_4 must satisfy (2.1)–(2.5). Hence (a_1, a_2, a_4) is one of the 28 triples given in Theorem 2.3. For each of the 28 triples (a_1, a_2, a_4) in Theorem 2.3, Theorem 2.4 ensures that $A_1, A_2, A_4, B_1, B_2, B_4, C$ are uniquely determined by

$$\begin{aligned} \eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z) &= A_1\xi_{12}(z) + A_2\xi_{12}(2z) + A_4\xi_{12}(4z) \\ &\quad + B_1\eta^{24}(z) + B_2\eta^{24}(2z) + B_4\eta^{24}(4z) + C\eta^8(z)\eta^{16}(4z). \end{aligned}$$

Expanding each of $\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_4}(4z), \xi_{12}(z), \ldots, \eta^8(z)\eta^{16}(4z)$ in powers of $e^{2\pi i z}$ and equating the coefficients of $e^{2\pi i n z}$ $(n = 0, 1, 2, 3, 4, 5, 6)$, we obtain seven linear equations for the seven quantities $A_1, A_2, \ldots, C$. From these we obtain the values of $A_1, A_2, \ldots, C$ given in Theorem 1.1. □

3. Final Remarks

The second author has recently given the following arithmetic formula for $\tau(n)$, namely,

$$\tau(n) = \frac{1}{4} \sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8 \\ x_1^2+\cdots+x_8^2=2n}} x_1^2 x_2^2 (x_1^2 - 3x_3^2)(x_2^2 - 3x_4^2),$$

as a special case of a general product-to-sum formula [16, Theorem 4.1]. As another application of this product-to-sum formula we obtain a formula for $\omega(n)$. In Theorem 1.1 of [16] we take $r = 0, s = 2, t = 0, u = 0, v = 2, w = 0, x = 0, y = 1$ so that $k = 2, l = 3, m = 8$. We obtain the following formula for $\omega(n)$.

Theorem 3.1. *Let* $n \in \mathbb{N}$ *satisfy* $n \geq 3$. *Then*

$$\omega(n) = \frac{1}{8} \sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8 \\ x_1^2+\cdots+x_4^2+2x_5^2+\cdots+2x_8^2=n}} (x_1^2 - 2x_7^2)(x_2^2 - 2x_8^2)(x_3^4 - 3x_3^2 x_4^2).$$

We conclude with the following property of $\omega(n)$.

Theorem 3.2. *Let* $n \in \mathbb{N}$. *Then*

$$\omega(n) = 0 \quad \textit{for } n \equiv 2 \pmod 4.$$

Proof. We recall the parameters x and z defined in [2, p. 120], namely,

$$x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z = \varphi^2(q).$$

From Berndt's catalogue of formulas [2, pp. 122–123] we deduce

$$E_1 = 2^{-1/6} q^{-1/24} x^{1/24} (1-x)^{1/6} z^{1/2},$$
$$E_2 = 2^{-1/3} q^{-1/12} x^{1/12} (1-x)^{1/12} z^{1/2},$$
$$E_4 = 2^{-2/3} q^{-1/6} x^{1/6} (1-x)^{1/24} z^{1/2}.$$

Hence

$$A(q) := q^3 E_1^{-8} E_2^{24} E_4^8 = 2^{-12} x^3 (1-x) z^{12}.$$

Jacobi's change of sign principle [3, p. 565] tells us that

$$x(-q) = \frac{x}{x-1}, \quad z(-q) = (1-x)^{1/2} z,$$

so

$$A(-q) = -2^{-12} x^3 (1-x)^2 z^{12}.$$

Thus

$$A(q) + A(-q) = 2^{-12} x^4 (1-x) z^{12}. \tag{3.1}$$

Cheng's rotation principle [3, p. 565] asserts

$$x(iq) = \frac{-8i(1-x)^{1/4}(1-(1-x)^{1/2})}{(1-i(1-x)^{1/4})^4}, \quad z(iq) = \frac{i}{2}(1-i(1-x)^{1/4})^2 z. \quad (3.2)$$

Mapping q to iq in (3.1), and appealing to (3.2), we find

$$A(iq) + A(-iq) = 2^{-12}x^4(1-x)z^{12}.$$

Thus $A(q) - A(iq) + A(-q) - A(-iq) = 0$. Set $a_n = [A(q)]_n$. Then

$$\sum_{\substack{n=0 \\ n \equiv 2 \pmod 4}}^{\infty} a_n q^n = \frac{1}{4}\sum_{n=0}^{\infty} a_n(1 - i^n + (-1)^n - (-i)^n)q^n = 0.$$

Hence

$$\omega(n) = [q^3 E_1^{-8} E_2^{24} E_4^8]_n = [A(q)]_n = a_n = 0 \quad \text{for } n \equiv 2 \pmod 4,$$

as claimed. □

Theorems 1.2, 1.3, 1.5, 1.8, and 3.2 show that the number of representations $(x_1, \ldots, x_{16}, x_{17}, \ldots, x_{24}) \in \mathbb{Z}^{16} \times \mathbb{N}_0^8$ of $n-1$ as

$$x_1^2 + \cdots + x_{16}^2 + \frac{x_{17}(x_{17}+1)}{2} + \cdots + \frac{x_{24}(x_{24}+1)}{2},$$

the number of representations $(x_1, \ldots, x_8, x_9, \ldots, x_{24}) \in \mathbb{Z}^8 \times \mathbb{N}_0^{16}$ of $n-2$ as

$$x_1^2 + \cdots + x_8^2 + \frac{x_9(x_9+1)}{2} + \cdots + \frac{x_{24}(x_{24}+1)}{2},$$

the number of representations $(x_1, \ldots, x_{24}) \in \mathbb{N}_0^{24}$ of $n-4$ as

$$\frac{x_1(x_1+1)}{2} + \cdots + \frac{x_{16}(x_{16}+1)}{2} + x_{17}(x_{17}+1) + \cdots + x_{24}(x_{24}+1),$$

and the number of representations $(x_1, \ldots, x_{24}) \in \mathbb{N}_0^{24}$ of $n-5$ as

$$\frac{x_1(x_1+1)}{2} + \cdots + \frac{x_8(x_8+1)}{2} + x_9(x_9+1) + \cdots + x_{24}(x_{24}+1),$$

depend only upon $\sigma_{11}(n), \sigma_{11}(n/2), \tau(n)$ and $\tau(n/2)$ when $n \equiv 2 \pmod 4$.

In this paper we have considered certain eta quotients in the space $M_{12}(\Gamma_0(4))$. In [17] the second author has essentially considered similar eta quotients in the space $M_2(\Gamma_0(4))$ but in an entirely different manner.

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