

EULER PRODUCTS IN RAMANUJAN'S LOST NOTEBOOK

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In his famous paper, “On certain arithmetical functions”, Ramanujan offers for the first time the Euler product of the Dirichlet series in which the coefficients are given by Ramanujan’s tau-function. In his lost notebook, Ramanujan records further Euler products for L -series attached to modular forms, and, typically, does not record proofs for these claims. In this semi-expository article, for the Euler products appearing in his lost notebook, we provide or sketch proofs using elementary methods, binary quadratic forms, and modular forms.

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1. Introduction

In his famous paper, *On certain arithmetical functions* [22], Ramanujan offers for the first time the Euler product for what is now known as Ramanujan’s Dirichlet series. More precisely, if $\tau(n)$ denotes Ramanujan’s tau-function, then [22, p. 153]

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}, \quad \sigma = \operatorname{Re} s > \frac{13}{2}, \quad (1.1)$$

where the product is over all primes p . It should be emphasized that (1.1) is prefaced by the words, “For it appears that”. Thus, at the time he wrote [22], Ramanujan did not have a proof of (1.1), and we are uncertain if he later devised a proof or not. In the same article, Ramanujan states further Euler products for the Dirichlet series associated with special cases of

$$q\{(1 - q^{24/\alpha})(1 - q^{48/\alpha})(1 - q^{72/\alpha}) \cdots\}^\alpha =: \sum_{n=1}^{\infty} \Psi_\alpha(n)q^n, \quad \alpha|24,$$

for which he also does not provide proofs. Proofs were given by Mordell in 1917 [17, p. 121].

Published *with* the lost notebook is a more complete list of Ramanujan’s discoveries about such Euler products [24, pp. 233–235]. In particular, in his paper [22], Ramanujan examines only Euler products corresponding to powers of the Dedekind eta-function, while in the manuscript in [24], Ramanujan examines Dirichlet series arising from powers of the eta-function multiplied by certain Eisenstein series. This list of 46 modular forms with their corresponding Euler products is examined in the last section of the present paper. It is to be emphasized that in this list, the associated modular form contains a power of *only one* eta-function.

At scattered places *in* his lost notebook, Ramanujan offers further examples of Euler products that we relate below. Most of these were claimed to have been proven by Rangachari [25] using the theory of modular forms, but his proofs are incomplete, and he failed to notice that several of Ramanujan’s claims need corrections. See also Rangachari’s paper [26] for a discussion of some of these results. Raghavan [21] disproved one of Ramanujan’s claims, but did not offer a corrected version. Being in the original lost notebook, these scattered results were most likely discovered during the last year of Ramanujan’s life, after he had returned to India. Thus, from time to time, he was clearly seeking further theorems along the lines of what he wrote in [22], and it is unfortunate that he did not live longer to further develop his ideas.

The results are dispersed somewhat randomly, and we shall examine them in the order in which they appear in [24]. After we examine the scattered claims, we focus our attention on the list given in [24, pp. 233–235]. It is doubtful that few, if any, of our arguments coincide with those found by Ramanujan. In particular, we use ideas, for example, from the theory of modular forms, with which Ramanujan would have been unfamiliar. We have attempted to present proofs that are as elementary as possible, but even these proofs are unlikely to be close to any found by Ramanujan. It would be of enormous interest if one could discern how Ramanujan discovered these beautiful theorems on Euler products.

In our accounts that follow, we employ Ramanujan’s notations for theta functions and Eisenstein series. As usual, set

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

Ramanujan's function $f(-q)$ is defined by

$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} = (q; q)_{\infty} =: q^{-1/24} \eta(z), \quad q = e^{2\pi iz}, z \in \mathbb{H}, \quad (1.2)$$

where the second equality above is the pentagonal number theorem, where $\eta(z)$ denotes the Dedekind eta-function, and where $\mathbb{H} = \{z : \text{Im } z > 0\}$. Ramanujan's Eisenstein series are defined for $|q| < 1$ by

$$P(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \quad (1.3)$$

$$Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad (1.4)$$

$$R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \quad (1.5)$$

2. Scattered Entries on Euler Products

Entry 2.1 (p. 54). *If $f(-q)$ is defined by (1.2) and*

$$\sum_{n=1}^{\infty} a_n q^n := q f^3(-q) f^3(-q^7), \quad (2.1)$$

then

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{1 + 7^{1-s}} \prod_p \frac{1}{1 - p^{2(1-s)}} \prod_q \frac{1}{1 + 2c_q q^{-s} + q^{2(1-s)}}, \quad (2.2)$$

where the first product is over all primes $p \equiv 3, 5, 6 \pmod{7}$, the second product is over all primes $q \equiv 1, 2, 4 \pmod{7}$, and

$$c_q = \begin{cases} \frac{3}{2}, & \text{if } q = 2, \\ 7v^2 - u^2, & \text{if } q = u^2 + 7v^2. \end{cases} \quad (2.3)$$

Entry 2.1 was essentially established by Rangachari [25]; see formula (5) under (b) in his paper. However, like Ramanujan, he failed to see that c_2 had to be defined separately from the remaining cases when $q \equiv 1, 2, 4 \pmod{7}$.

Ramanujan records another form of Entry 2.1 in his manuscript on the partition and tau-functions; in particular, see [24, p. 146] or [1, p. 105].

Entry 2.2 (p. 146). *Define the coefficients a_n , $n \geq 1$, by (2.1). Then*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n}{n^s} &= \frac{1}{1 + 7^{1-s}} \prod_{p \equiv 3, 5, 6 \pmod{7}} \frac{1}{1 - p^{2(1-s)}} \\ &\times \prod_{p \equiv 1, 2, 4 \pmod{7}} \frac{1}{1 + C_p p^{-s} + p^{2(1-s)}}, \end{aligned} \quad (2.4)$$

where

$$C_p = 2p - a^2 \tag{2.5}$$

with

$$4p = a^2 + 7b^2. \tag{2.6}$$

If p is odd, then the equality $p = u^2 + 7v^2$ implies that $4p = (2u)^2 + 7(2v)^2$. Conversely, if $4p = a^2 + 7b^2$ and p is odd, then a and b are even and $p = (a/2)^2 + 7(b/2)^2$. Hence, (2.2) is equivalent to (2.4) when p is odd, and, in particular,

$$C_p = 2p - a^2 = 2(p - 2u^2) = 2(u^2 + 7v^2 - 2u^2) = 2(7v^2 - u^2) = 2c_p.$$

When p is even, it is easy to check that C_2 is equal to $2c_2$. We thus see that Entries 2.1 and 2.2 are equivalent.

Entry 2.2 was discussed by Berndt and Ono in their paper [2, Eq. (8.4)], and they remarked that it could be proved with two applications of Jacobi’s identity

$$(q; q)_\infty^3 = \sum_{n=0}^\infty (-1)^n (2n + 1) q^{n(n+1)/2}, \tag{2.7}$$

but they did not supply a complete proof; see also [1, p. 145]. The first complete proof was given by Chan, Cooper and Liaw [3], and it is their proof that we now give below.

Proof of (2.3). As indicated in [2, 25], using the Dedekind eta-function $\eta(z)$, which is defined in (1.2), we see that

$$F(z) := \sum_{n=1}^\infty a_n q^n = \eta^3(z) \eta^3(7z) = q f^3(-q) f^3(-q^7), \quad q = e^{2\pi iz}, \quad z \in \mathbb{H},$$

is in $\mathcal{S} := S_3(\Gamma_0(7), (\frac{\cdot}{7}))$, the space of weight 3 cusp forms on $\Gamma_0(7)$ with the Legendre symbol $(\frac{\cdot}{7})$ as the character, which can be deduced by applying Newman’s criterion for η -products [19]. The space \mathcal{S} is one dimensional [5, Théorème 1], and hence $F(z)$ is an eigenform. Consequently, the corresponding Dirichlet series has an Euler product expansion [14, p. 163]

$$\sum_{n=1}^\infty \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s} + \left(\frac{p}{7}\right) p^{2(1-s)}}. \tag{2.8}$$

It remains to determine a_p for all primes p .

We write Jacobi’s identity (2.7) in the form

$$\eta^3(z) = \sum_{\substack{\alpha=-\infty \\ \alpha \equiv 1 \pmod{4}}}^\infty \alpha q^{\alpha^2/8}. \tag{2.9}$$

Therefore,

$$\eta^3(z)\eta^3(7z) = \sum_{\substack{\alpha, \beta = -\infty \\ \alpha, \beta \equiv 1 \pmod{4}}}^{\infty} \alpha\beta q^{(\alpha^2+7\beta^2)/8}. \tag{2.10}$$

Hence, for all positive integers n , we have, from (2.1) and (2.10),

$$a_n = \sum_{\substack{\alpha, \beta = -\infty \\ (\alpha, \beta) \equiv (1, 1) \pmod{4} \\ 8n = \alpha^2 + 7\beta^2}}^{\infty} \alpha\beta. \tag{2.11}$$

First, we take $n = 2$ in (2.11). The only pair $(\alpha, \beta) \in \mathbb{Z}^2$ satisfying $16 = \alpha^2 + 7\beta^2$ and $(\alpha, \beta) \equiv (1, 1) \pmod{4}$ is $(\alpha, \beta) = (-3, 1)$. Hence, $a_2 = -3$. This gives the value of c_2 in (2.3).

Second, we take $n = 7$ in (2.11). The only pair $(\alpha, \beta) \in \mathbb{Z}^2$ satisfying $56 = \alpha^2 + 7\beta^2$ and $(\alpha, \beta) \equiv (1, 1) \pmod{4}$ is $(\alpha, \beta) = (-7, 1)$. Hence, $a_7 = -7$. This gives the first factor in (2.2).

Now let n denote a prime $p \equiv 3, 5, 6 \pmod{7}$, so that $\left(\frac{-7}{p}\right) = -1$, in (2.11), where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol. In this case, $8p \neq \alpha^2 + 7\beta^2$ for any odd integers α and β , for otherwise, $8p = \alpha^2 + 7\beta^2$ and thus

$$\left(\frac{-7}{p}\right) = \left(\frac{-7\beta^2}{p}\right) = \left(\frac{\alpha^2 - 8p}{p}\right) = \left(\frac{\alpha^2}{p}\right) = 1,$$

which is a contradiction. Thus, $a_p = 0$ for $p \equiv 3, 5, 6 \pmod{7}$, and we obtain the second product on the right-hand side of (2.2).

Finally, we let n denote a prime $p \equiv 1, 2, 4 \pmod{7}$, with $p \neq 2$, in (2.11). In this case there exist integers A and B such that $p = A^2 + 7B^2$ [10, Exercise 8, p. 309]. Since p is odd, all such pairs (A, B) satisfy $A + B \equiv 1 \pmod{2}$. The mapping $(A, B) \mapsto (-A, -B)$ shows that half of these pairs satisfy $A + B \equiv 1 \pmod{4}$, with the other half satisfying $A + B \equiv 3 \pmod{4}$. Let

$$S := \{(A, B) \in \mathbb{Z}^2 : p = A^2 + 7B^2, A + B \equiv 1 \pmod{4}\}$$

and

$$T := \{(\alpha, \beta) \in \mathbb{Z}^2 : 8p = \alpha^2 + 7\beta^2, \alpha \equiv \beta \equiv 1 \pmod{4}\}.$$

We note that if $(\alpha, \beta) \in T$, then $\alpha \equiv \beta \pmod{8}$. The mapping $\lambda : S \mapsto T$ given by

$$\lambda((A, B)) = (A - 7B, A + B) \tag{2.12}$$

is a bijection. Hence, applying (2.12) in (2.11), we obtain

$$\begin{aligned} a_p &= \sum_{(\alpha, \beta) \in T} \alpha\beta = \sum_{(A, B) \in S} (A - 7B)(A + B) \\ &= \sum_{\substack{(A, B) \in \mathbb{Z}^2 \\ p = A^2 + 7B^2 \\ A + B \equiv 1 \pmod{4}}} (A - 7B)(A + B) = \frac{1}{2} \sum_{\substack{(A, B) \in \mathbb{Z}^2 \\ p = A^2 + 7B^2}} (A - 7B)(A + B). \end{aligned} \tag{2.13}$$

If $(A, B) \in \mathbb{Z}^2$ is any solution of $p = A^2 + 7B^2$, all the solutions are $(A, B), (A, -B), (-A, B), (-A, -B)$, so that by (2.13),

$$a_p = \frac{1}{2}((A - 7B)(A + B) + (A + 7B)(A - B) + (-A - 7B)(-A + B) + (-A + 7B)(-A - B)) = 2(A^2 - 7B^2).$$

This completes the proof of (2.3) and the derivation of the third factor in (2.2) for primes $p \neq 2$. □

Entry 2.3 (p. 207). *If*

$$\sum_{n=1}^{\infty} q(n)q^n := qf^4(-q)f^4(-q^5), \tag{2.14}$$

then

$$\sum_{n=1}^{\infty} \frac{q(n)}{n^s} = \frac{1}{1 + 5^{1-s}} \prod_p \frac{1}{1 - q(p)p^{-s} + p^{3-2s}}, \tag{2.15}$$

where the product is over all primes p except $p = 5$. Furthermore, $q(2) = -4, q(3) = 2, q(5) = -5, q(7) = 6$, and generally $q^2(p) < 4p^3$.

Entry 2.3 was not discussed by Rangachari [25]. However, it does fall under the theory outlined in his paper. We have corrected a misprint in the lost notebook [24]; Ramanujan had written p^{3-s} instead of p^{3-2s} in the last term in the denominator on the right-hand side of (2.15). The values of $q(n), n = 2, 3, 5, 7$, calculated by Ramanujan are correct. It is doubtful that Ramanujan had a proof of the inequality $q^2(p) < 4p^3$, which follows from the deep work of Deligne [6]. It would be extremely interesting to know how Ramanujan deduced it. At the end of our proof of Entry 2.3, we provide an elementary proof of a much weaker result.

Proof of Entry 2.3. The coefficients $q(n), n \geq 1$, are defined by (2.14). By a paper by Martin [16, Table 1, p. 4852], $q(n)$ is a multiplicative function of n . From a paper of Newman [18, p. 487], $q(p^\alpha)$, where p is a prime not equal to 5, satisfies the recurrence relation

$$q(p^\alpha) - q(p)q(p^{\alpha-1}) + p^3q(p^{\alpha-2}) = 0, \quad \alpha \in \mathbb{N}, \alpha \geq 2, \tag{2.16}$$

and

$$q(5^\alpha) = (-5)^\alpha, \quad \alpha \in \mathbb{N}_0. \tag{2.17}$$

Hence, for $p \neq 5$, by (2.16),

$$\begin{aligned} & (1 - q(p)p^{-s} + p^{3-2s}) \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}} \\ &= \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}} - q(p) \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{(\alpha+1)s}} + p^3 \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{(\alpha+2)s}} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}} - q(p) \sum_{\alpha=1}^{\infty} \frac{q(p^{\alpha-1})}{p^{\alpha s}} + p^3 \sum_{\alpha=2}^{\infty} \frac{q(p^{\alpha-2})}{p^{\alpha s}} \\
 &= q(1) + \left(\frac{q(p)}{p^s} - \frac{q(p)q(1)}{p^s} \right) + \sum_{\alpha=2}^{\infty} \frac{q(p^\alpha) - q(p)q(p^{\alpha-1}) + p^3 q(p^{\alpha-2})}{p^{\alpha s}} \\
 &= 1.
 \end{aligned} \tag{2.18}$$

It follows from (2.18) that

$$\sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}} = \frac{1}{1 - q(p)p^{-s} + p^{3-2s}}, \quad p \neq 5. \tag{2.19}$$

Also, from (2.17),

$$\sum_{\alpha=0}^{\infty} \frac{q(5^\alpha)}{5^{\alpha s}} = \sum_{\alpha=0}^{\infty} (-5^{1-s})^\alpha = \frac{1}{1 + 5^{1-s}}. \tag{2.20}$$

Thus, as $q(n)$ is multiplicative, we deduce from (2.19) and (2.20) that

$$\sum_{n=1}^{\infty} \frac{q(n)}{n^s} = \sum_{\alpha=0}^{\infty} \frac{q(5^\alpha)}{5^{\alpha s}} \prod_{p \neq 5} \sum_{\alpha=0}^{\infty} \frac{q(p^\alpha)}{p^{\alpha s}} = \frac{1}{1 + 5^{1-s}} \prod_{p \neq 5} \frac{1}{1 - q(p)p^{-s} + p^{3-2s}},$$

and so the proof of Entry 2.3 is complete, except for the inequality for $q(p)$. □

To obtain an elementary bound for $q(n)$, we use Ramanujan's two famous identities [1, p. 98]

$$\frac{(q; q)_\infty^5}{(q^5; q^5)_\infty} = 1 - 5 \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{5}{d} \right) d \right) q^n$$

and

$$q \frac{(q^5; q^5)_\infty^5}{(q; q)_\infty} = \sum_{m=1}^{\infty} \left(\sum_{e|m} \left(\frac{5}{m/e} \right) e \right) q^m,$$

where $\left(\frac{\cdot}{n} \right)$ denotes the Kronecker symbol, to deduce that

$$\begin{aligned}
 &q(q; q)_\infty^4 (q^5; q^5)_\infty^4 \\
 &= \sum_{m=1}^{\infty} \left(\sum_{e|m} \left(\frac{5}{m/e} \right) e \right) q^m - 5 \sum_{N=1}^{\infty} \left(\sum_{\substack{m, n=1 \\ m+n=N}}^{\infty} \sum_{d|n} \sum_{e|m} \left(\frac{5}{dm/e} \right) de \right) q^N.
 \end{aligned}$$

Hence, writing $d = a$, $e = b$, $m = by$, and $n = ax$, we deduce that, for $N \geq 1$,

$$q(N) = \sum_{d|N} \left(\frac{5}{N/d} \right) d - 5 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N}} \left(\frac{5}{ay} \right) ab.$$

Thus, using below an evaluation from [22; 23, Table IV, no. 1, p. 146], we find that

$$\begin{aligned}
 |q(N)| &< \sum_{d|N} d + 5 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N}} ab \\
 &= \sigma(N) + 5 \sum_{r=1}^{N-1} \sigma(r)\sigma(N-r) \\
 &= \sigma(N) + 5 \left(\frac{5}{12}\sigma_3(N) + \left(\frac{1}{12} - \frac{1}{2}N \right) \sigma(N) \right) \\
 &= \frac{25}{12}\sigma_3(N) + \left(\frac{17}{12} - \frac{5}{2}N \right) \sigma(N) \\
 &< \frac{25}{12}\sigma_3(N).
 \end{aligned}$$

In particular, if $N = p$ is a prime,

$$|q(p)| < \frac{25}{12}\sigma_3(p) = \frac{25}{12}(1^3 + p^3) < \frac{25}{6}p^3,$$

which is much weaker than Ramanujan’s assertion.

Entry 2.4 (p. 247). *If*

$$\sum_{n=1}^{\infty} \phi(n)q^n := qf^{12}(-q^2), \tag{2.21}$$

then

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \prod_p \frac{1}{1 - \phi(p)p^{-s} + p^{5-2s}}, \tag{2.22}$$

where the product is over all odd primes p .

Entry 2.4 is actually a special case of a general claim made without proof in Ramanujan’s paper [22, p. 162]. Entry 2.4 was proved by Mordell [17, p. 121]; it is the case $a = 6$ in Mordell’s paper. A proof was also given by Rangachari [25].

Entry 2.5 (p. 247). *If*

$$\sum_{n=1}^{\infty} \phi(n)q^n := qf(-q^2)f(-q^{22}), \tag{2.23}$$

then

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{1}{1 - 11^{-s}} \prod_p \frac{1}{1 + p^{-s} + p^{-2s}} \prod_q \frac{1}{1 - q^{-2s}} \prod_r \frac{1}{(1 - r^{-s})^2}, \tag{2.24}$$

where the first product is over all primes p such that $p \equiv 1, 3, 4, 5, 9 \pmod{11}$, the second product is over all odd primes q such that $q \equiv 2, 6, 7, 8, 10 \pmod{11}$, and

the third product is over all primes r such that r can be written in the form $r = 11A^2 + B^2$.

Entry 2.5 is formula (2) under part (b) in Rangachari's paper. Unfortunately, Rangachari failed to notice that Entry 2.5 is incorrect, and so therefore is his claim to a proof. Entry 2.5 was first proved in its corrected form by Sun and Williams [30, Theorem 7.2, p. 386].

Entry 2.6 (p. 247). *If*

$$\sum_{n=1}^{\infty} \phi(n)q^n := qf(-q^3)f(-q^{21}), \tag{2.25}$$

then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} &= \frac{1}{1+7^{-s}} \prod_p \frac{1}{1+p^{-2s}} \prod_q \frac{1}{1-q^{-2s}} \\ &\times \prod_r \frac{1}{(1+r^{-s})^2} \prod_t \frac{1}{(1-t^{-s})^2}, \end{aligned} \tag{2.26}$$

where the first product is over all odd primes p such that $p \equiv 2, 8, 11 \pmod{21}$, the second product is over all primes q such that $q \equiv 5, 17, 20 \pmod{21}$, the third product is over all primes r such that r can be written in the form $r = 9A^2 + 7B^2$, and the fourth product is over all primes t such that t can be written in the form $t = A^2 + 63B^2$.

Entry 2.6 is formula (1) under part (b) in Rangachari's paper [25]. Entry 2.6 is incorrect, and so therefore is Rangachari's proof. The first proof of a corrected version of Entry 2.6 was given by Sun and Williams [30, Theorem 8.2(i), p. 388]. The proofs of corrected versions of Entries 2.5 and 2.6 by Sun and Williams are discussed in more detail in Sec. 3, which is devoted to their methods.

Entry 2.7 (p. 249). *Define*

$$\sum_{n=1}^{\infty} \phi(n)q^n := qf^4(-q^6)R(q^6), \tag{2.27}$$

where $R(q)$ is defined in (1.5). Then

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \prod_p \frac{1}{1 - \phi(p)p^{-s} + p^{7-2s}}, \tag{2.28}$$

where the product is over all primes p exceeding 3 and

$$\phi(p) = \begin{cases} 0, & \text{if } p \equiv -1 \pmod{6}, \\ (A + iB\sqrt{3})^7 + (A - iB\sqrt{3})^7, & \text{if } p \equiv 1 \pmod{6}. \end{cases} \tag{2.29}$$

where A and B are determined by $p = A^2 + 3B^2$.

Entry 2.7 can also be found under List I on pp. 233–235 of [24] or in List I in Sec. 4 below.

In Ramanujan’s formulation, he wrote “where A and B are the same as before.”

Proof of Entry 2.7. Let

$$\omega = \frac{1 + \sqrt{-3}}{2}.$$

For the convenience of future calculations, we record the values

$$\begin{aligned} \omega^2 &= \frac{-1 + \sqrt{-3}}{2} = -1 + \omega, & \omega^4 &= \frac{-1 - \sqrt{-3}}{2} = -\omega, & \omega^5 &= \frac{1 - \sqrt{-3}}{2} = 1 - \omega, \\ \sqrt{-3} &= -1 + 2\omega, & \omega\sqrt{-3} &= -2 + \omega, & \omega^2\sqrt{-3} &= -1 - \omega. \end{aligned}$$

We briefly review basic facts about the ring of integers $\mathbb{Z} + \mathbb{Z}\omega = \{x + y\omega \mid x, y \in \mathbb{Z}\}$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$, which has discriminant -3 . It is well known that $\mathbb{Z} + \mathbb{Z}\omega$ is a unique factorization domain. The group of units in $\mathbb{Z} + \mathbb{Z}\omega$ is the cyclic group of order 6 generated by ω . The Eisenstein integer $2 + 2\omega$ is the product of two irreducible integers, namely, 2 and $1 + \omega$.

We now define a character χ on $\mathbb{Z} + \mathbb{Z}\omega$ modulo $2 + 2\omega$. Let $x + y\omega \in \mathbb{Z} + \mathbb{Z}\omega$. First, observe that

$$\begin{aligned} 2 \mid \gcd(x + y\omega, 2 + 2\omega) &\Leftrightarrow x \equiv y \equiv 0 \pmod{2}, \\ (1 + \omega) \mid \gcd(x + y\omega, 2 + 2\omega) &\Leftrightarrow x \equiv y \pmod{3}. \end{aligned}$$

Hence,

$$\begin{aligned} \gcd(x + y\omega, 2 + 2\omega) &= 1 \\ &\Leftrightarrow (x, y) \equiv (0, 1), (1, 0), \text{ or } (1, 1) \pmod{2} \text{ and } x \not\equiv y \pmod{3}. \end{aligned}$$

For those $x + y\omega$ coprime with $2 + 2\omega$,

$$\begin{aligned} x + y\omega &\equiv 1 \pmod{2 + 2\omega}, & \text{if } (x, y) &\equiv (1, 0) \pmod{2} \text{ and } x - y \equiv 1 \pmod{3}, \\ x + y\omega &\equiv \omega \pmod{2 + 2\omega}, & \text{if } (x, y) &\equiv (0, 1) \pmod{2} \text{ and } x - y \equiv 2 \pmod{3}, \\ x + y\omega &\equiv \omega^2 \pmod{2 + 2\omega}, & \text{if } (x, y) &\equiv (1, 1) \pmod{2} \text{ and } x - y \equiv 1 \pmod{3}, \\ x + y\omega &\equiv \omega^3 \pmod{2 + 2\omega}, & \text{if } (x, y) &\equiv (1, 0) \pmod{2} \text{ and } x - y \equiv 2 \pmod{3}, \\ x + y\omega &\equiv \omega^4 \pmod{2 + 2\omega}, & \text{if } (x, y) &\equiv (0, 1) \pmod{2} \text{ and } x - y \equiv 1 \pmod{3}, \\ x + y\omega &\equiv \omega^5 \pmod{2 + 2\omega}, & \text{if } (x, y) &\equiv (1, 1) \pmod{2} \text{ and } x - y \equiv 2 \pmod{3}. \end{aligned}$$

Hence, we can define a character χ of order 6 on $\mathbb{Z} + \mathbb{Z}\omega \pmod{2 + 2\omega}$ by

$$\chi(x + y\omega) = \begin{cases} \omega^{-\ell}, & \text{if } x + y\omega \equiv \omega^\ell \pmod{2 + 2\omega} \text{ for some } \ell \in \{0, 1, 2, 3, 4, 5\}, \\ 0, & \text{if } \gcd(x + y\omega, 2 + 2\omega) \neq 1. \end{cases}$$

In particular, note that

$$\chi(\omega) = \omega^{-1} = \bar{\omega}.$$

If ϵ is a unit in $\mathbb{Z} + \mathbb{Z}\omega$, then $\epsilon = \omega^\ell$ for some $\ell \in \{0, 1, 2, 3, 4, 5\}$ and

$$\chi(\epsilon)\epsilon^7 = \chi(\omega^\ell)\omega^{7\ell} = \chi^\ell(\omega)\omega^{7\ell} = (\omega^{-1})^\ell\omega^\ell = 1.$$

Thus, by [15, Eqs. (5.8) and (5.9)], the Hecke theta series $\theta_8(-3, \chi, z)$ is given by

$$\theta_8(-3, \chi, z) = \frac{1}{6} \sum_{x+y\omega \in \mathbb{Z} + \mathbb{Z}\omega} \chi(x+y\omega)(x+y\omega)^7 e^{2\pi i(x+y\omega)(x+y\bar{\omega})z}, \tag{2.30}$$

where $z \in \mathbb{H}$. Noting that $(x+y\omega)(x+y\bar{\omega}) = x^2 + xy + y^2$, we define, for each $\ell \in \{0, 1, 2, 3, 4, 5\}$,

$$A_\ell := \omega^{-\ell} \sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ x+y\omega \equiv \omega^\ell \pmod{2+2\omega}}} (x+y\omega)^7 q^{x^2+xy+y^2}, \quad q = e^{2\pi iz}.$$

Then,

$$\begin{aligned} \frac{1}{6} \sum_{\ell=0}^5 A_\ell &= \frac{1}{6} \sum_{\ell=0}^5 \omega^{-\ell} \sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ x+y\omega \equiv \omega^\ell \pmod{2+2\omega}}} (x+y\omega)^7 q^{x^2+xy+y^2} \\ &= \frac{1}{6} \sum_{\ell=0}^5 \sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ x+y\omega \equiv \omega^\ell \pmod{2+2\omega}}} \chi(x+y\omega)(x+y\omega)^7 q^{x^2+xy+y^2} \\ &= \frac{1}{6} \sum_{\substack{x+y\omega \in \mathbb{Z} + \mathbb{Z}\omega \\ \gcd(x+y\omega, 2+2\omega)=1}} \chi(x+y\omega)(x+y\omega)^7 q^{x^2+xy+y^2} \\ &= \frac{1}{6} \sum_{x+y\omega \in \mathbb{Z} + \mathbb{Z}\omega} \chi(x+y\omega)(x+y\omega)^7 e^{2\pi i(x+y\omega)(x+y\bar{\omega})z} \\ &= \theta_8(-3, \chi, z), \end{aligned} \tag{2.31}$$

by (2.30).

Next, we evaluate A_0, A_1, \dots, A_5 by making the indicated changes of variable in the corresponding series:

$$\begin{aligned} A_0: & (x, y) = (r - s, 2s), \\ A_1: & (x, y) = (2s, r - s), \\ A_2: & (x, y) = (-r + s, r + s), \\ A_3: & (x, y) = (-r - s, 2s), \\ A_4: & (x, y) = (2s, -r - s), \\ A_5: & (x, y) = (r + s, -r + s). \end{aligned}$$

We calculate one of the sums; the other calculations are similar. To that end,

$$\begin{aligned}
 A_5 &= \omega^{-5} \sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ (x,y) \equiv (1,1) \pmod{2} \\ x-y \equiv 2 \pmod{3}}} (x+y\omega)^7 q^{x^2+xy+y^2} \\
 &= \omega \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{3} \\ r \equiv 1 \pmod{2}}} (r(1-\omega) + s(1+\omega))^7 q^{r^2+3s^2} \\
 &= \omega \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{3} \\ r \equiv 1 \pmod{2}}} (-r\omega^2 - s\omega^2\sqrt{-3})^7 q^{r^2+3s^2} \\
 &= \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{3} \\ r \equiv 1 \pmod{2}}} (r + s\sqrt{-3})^7 q^{r^2+3s^2}.
 \end{aligned}$$

In summary, we find that

$$A_0 = A_4 = A_5 = \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{2} \\ r \equiv 1 \pmod{3}}} (r + s\sqrt{-3})^7 q^{r^2+3s^2}, \tag{2.32}$$

$$A_1 = A_2 = A_3 = \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{2} \\ r \equiv 1 \pmod{3}}} (r - s\sqrt{-3})^7 q^{r^2+3s^2}. \tag{2.33}$$

Hence, by (2.32) and (2.33),

$$\sum_{\ell=0}^5 A_\ell = 3 \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{2} \\ r \equiv 1 \pmod{3}}} ((r + s\sqrt{-3})^7 + (r - s\sqrt{-3})^7) q^{r^2+3s^2},$$

and so, by (2.31),

$$\theta_8(-3, \chi, z) = \frac{1}{2} \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{2} \\ r \equiv 1 \pmod{3}}} ((r + s\sqrt{-3})^7 + (r - s\sqrt{-3})^7) q^{r^2+3s^2}. \tag{2.34}$$

Now [15, p. 122],

$$\theta_8(-3, \chi, z) = qf^4(-q^6)R(q^6). \tag{2.35}$$

Thus, from (2.27), (2.35), and (2.34),

$$\phi(n) = \frac{1}{2} \sum_{\substack{(r,s) \in \mathbb{Z}^2 \\ r-s \equiv 1 \pmod{2} \\ r \equiv 1 \pmod{3} \\ r^2+3s^2=n}} ((r + s\sqrt{-3})^7 + (r - s\sqrt{-3})^7). \tag{2.36}$$

As the only solution to $r^2 + 3s^2 = 1$, $r - s \equiv 1 \pmod{2}$, $r \equiv 1 \pmod{3}$ is $(r, s) = (1, 0)$, we find from (2.36) that

$$\phi(1) = 1. \tag{2.37}$$

The conditions $r - s \equiv 1 \pmod{2}$, $r^2 + 3s^2 = n$ imply that $n \equiv 1 \pmod{2}$. Hence,

$$\phi(n) = 0, \quad \text{if } 2|n. \tag{2.38}$$

The conditions $r \equiv 1 \pmod{3}$, $r^2 + 3s^2 = n$ imply that $n \equiv 1 \pmod{3}$. Hence,

$$\phi(n) = 0, \quad \text{if } 3|n. \tag{2.39}$$

If p is a prime with $p \equiv -1 \pmod{6}$ and n is odd, then there are no integers r and s such that $p^n = r^2 + 3s^2$. Thus,

$$\phi(p^n) = 0, \quad \text{if } p \equiv -1 \pmod{6}, \quad n \text{ odd}. \tag{2.40}$$

If p is a prime with $p \equiv 1 \pmod{6}$, then there are integers A and B such that $p = A^2 + 3B^2$. Replacing A by $-A$, if necessary, we may suppose that $A \equiv 1 \pmod{3}$. Replacing B by $-B$, if necessary, we may suppose that $B > 0$. All solutions of $p = x^2 + 3y^2$ are given by $(x, y) = (A, B), (A, -B), (-A, B), (-A, -B)$. Thus,

$$\begin{aligned} \phi(p) &= \frac{1}{2}(((A + B\sqrt{-3})^7 + (A - B\sqrt{-3})^7) + ((A - B\sqrt{-3})^7 + (A + B\sqrt{-3})^7)) \\ &= (A + B\sqrt{-3})^7 + (A - B\sqrt{-3})^7, \end{aligned} \tag{2.41}$$

where A and B are given uniquely by $p = A^2 + 3B^2$, $A \equiv 1 \pmod{3}$, $B > 0$.

The Hecke theta series

$$\theta_8(-3, \chi, z) = \sum_{n=1}^{\infty} \phi(n)e^{2\pi inz}, \tag{2.42}$$

where $\phi(n)$ is given by (2.36), is a modular form of weight 8 on the group $\Gamma_0(36)$ [15, p. 77]. We also know [15, p. 70] that $\phi(n)$ is multiplicative and satisfies the recursion relation

$$\phi(p^m) - \phi(p)\phi(p^{m-1}) + p^7\phi(p^{m-2}) = 0, \tag{2.43}$$

for each integer $m \geq 2$ and each prime p with $p \equiv 1 \pmod{6}$ [15, Eq. (5.7)], and the exact formula

$$\phi(p^{2m}) = (-p^7)^m, \tag{2.44}$$

for each positive integer m and each prime p with $p \equiv -1 \pmod{6}$ [15, p. 71].

Hence, for $p \equiv -1 \pmod{6}$, by (2.37), (2.40), and (2.44),

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} &= \sum_{m=0}^{\infty} \frac{\phi(p^{2m})}{p^{2ms}} = \sum_{m=0}^{\infty} \frac{(-p^7)^m}{p^{2ms}} \\ &= \sum_{m=0}^{\infty} (-p^{7-2s})^m = \frac{1}{1 + p^{7-2s}} = \frac{1}{1 - \phi(p)p^{-s} + p^{7-2s}}, \end{aligned} \tag{2.45}$$

by (2.40) once again.

For $p \equiv 1 \pmod{6}$, by (2.43),

$$\begin{aligned}
 & (1 - \phi(p)p^{-s} + p^{7-2s}) \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} \\
 &= \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} - \phi(p) \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{(m+1)s}} + p^7 \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{(m+2)s}} \\
 &= \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} - \phi(p) \sum_{m=1}^{\infty} \frac{\phi(p^{m-1})}{p^{ms}} + p^7 \sum_{m=2}^{\infty} \frac{\phi(p^{m-2})}{p^{ms}} \\
 &= 1 + \frac{\phi(p) - \phi(p)\phi(1)}{p^s} + \sum_{m=2}^{\infty} \frac{\phi(p^m) - \phi(p)\phi(p^{m-1}) + p^7\phi(p^{m-2})}{p^{ms}} \\
 &= 1.
 \end{aligned} \tag{2.46}$$

Thus, by (2.45) and (2.46),

$$\sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} = \frac{1}{1 - \phi(p)p^{-s} + p^{7-2s}}, \quad p > 3. \tag{2.47}$$

Clearly, by (2.38) and (2.39),

$$\sum_{m=0}^{\infty} \frac{\phi(2^m)}{2^{ms}} = 1 = \sum_{m=0}^{\infty} \frac{\phi(3^m)}{3^{ms}}, \tag{2.48}$$

respectively. Thus, as $\phi(n)$ is multiplicative, by (2.47) and (2.48),

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{\phi(n)}{n^s} &= \sum_{m=0}^{\infty} \frac{\phi(2^m)}{2^{ms}} \cdot \sum_{m=0}^{\infty} \frac{\phi(3^m)}{3^{ms}} \cdot \prod_{\substack{p>3 \\ p \text{ prime}}} \sum_{m=0}^{\infty} \frac{\phi(p^m)}{p^{ms}} \\
 &= \prod_{\substack{p>3 \\ p \text{ prime}}} \frac{1}{1 - \phi(p)p^{-s} + p^{7-2s}}.
 \end{aligned}$$

This completes the proof of Entry 2.7. □

Before discussing the next entry, we offer a remark on the convergence of the series and product in (2.28). The number of solutions $(x, y) \in \mathbb{Z}^2$ of $n = x^2 + 3y^2$ is $\ll_{\epsilon} n^{\epsilon}$, for each $\epsilon > 0$ [11, Corollary 9.1(a)]. Hence, by (2.36),

$$|\phi(n)| \ll_{\epsilon} n^{\frac{7}{2} + \epsilon},$$

for each $\epsilon > 0$. Therefore, the Dirichlet series on the left-hand side of (2.28) converges absolutely for $\text{Re } s > \frac{9}{2}$, and, moreover, we see that the product on the right-hand side of (2.28) also converges absolutely for $\text{Re } s > \frac{9}{2}$.

Entry 2.8 (p. 328). *If*

$$\sum_{n=1}^{\infty} a_n q^n := q^3 f^{18}(-q^4), \tag{2.49}$$

then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n}{n^s} &= \frac{1}{1 - 78 \cdot 3^{-s} + 3^{8-2s}} \frac{1}{1 + 510 \cdot 5^{-s} + 5^{8-2s}} \frac{1}{1 + 1404 \cdot 7^{-s} + 7^{8-2s}} \cdots \\ &\quad - \frac{1}{1 + 78 \cdot 3^{-s} + 3^{8-2s}} \frac{1}{1 - 510 \cdot 5^{-s} + 5^{8-2s}} \frac{1}{1 - 1404 \cdot 7^{-s} + 7^{8-2s}} \cdots \end{aligned} \tag{2.50}$$

Ramanujan writes a plus sign in front of 510 in each of the products above. Since one of the signs is likely to be incorrect, we have accordingly changed the second sign. Raghavan [21] numerically disproved Ramanujan's formula as he had written it. However, even with our slight change, Entry 2.8 is still incorrect. In searching for an appropriate linear combination of products of eta-functions and Eisenstein series in order to correct Entry 2.8, we are led to

$$78q^3 f^{18}(-q^4) + qf^6(-q^4)R(q^4) = q + 78q^3 - 510q^5 - 1404q^6 + \dots,$$

where $R(q)$ is defined in (1.5). From this calculation, it is possible that Ramanujan thought that $78q^3 f^{18}(-q^4) + qf^6(-q^4)R(q^4)$ has an Euler-product, which, however, is not the case. Nonetheless, Ramanujan later discovered the correct Euler product involving $q^3 f^{18}(-q^4)$, which is given in Entry 4.4.

We quote Ramanujan in the next entry. We emphasize that he does not provide any product representations.

Entry 2.9 (p. 328). *Presumably there are analogous results for $\sum_{n=1}^{\infty} a_n n^{-s}$ where $\sum_{n=1}^{\infty} a_n q^n$ is any of the functions*

$$\begin{aligned} &q^5 f^{10}(-q^{12}), \\ &q^7 f^{14}(-q^{12}), \\ &q^5 f^{20}(-q^6), \\ &q^{11} f^{22}(-q^{12}). \end{aligned}$$

Rangachari [25] established these Euler products, but they can also be found in Ramanujan's list of 46 products given in Sec. 4. Euler products corresponding to the first, second, and fourth functions above can be found in List IV, while the Euler product associated with the third modular form above can be found in List I.

Entry 2.10 (p. 329). *Define*

$$F(q) := qf^{16}(-q^3) =: \sum_{n=1}^{\infty} A_n q^n \tag{2.51}$$

and

$$F_1(q) := qf^8(-q^3)Q(q^3) =: \sum_{n=1}^{\infty} a_n q^n, \tag{2.52}$$

where $Q(q)$ is defined in (1.4). Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n + 6A_n\sqrt{10}}{n^s} &= \frac{1}{1 - 6\sqrt{10} \cdot 2^{-s} + 2^{7-2s}} \frac{1}{1 + 96\sqrt{10} \cdot 5^{-s} + 5^{7-2s}} \\ &\times \frac{1}{1 - 260 \cdot 7^{-s} + 7^{7-2s}} \frac{1}{1 + 1920\sqrt{10} \cdot 11^{-s} + 11^{7-2s}} \cdots \end{aligned} \tag{2.53}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n - 6A_n\sqrt{10}}{n^s} &= \frac{1}{1 + 6\sqrt{10} \cdot 2^{-s} + 2^{7-2s}} \frac{1}{1 - 96\sqrt{10} \cdot 5^{-s} + 5^{7-2s}} \\ &\times \frac{1}{1 - 260 \cdot 7^{-s} + 7^{7-2s}} \frac{1}{1 - 1920\sqrt{10} \cdot 11^{-s} + 11^{7-2s}} \cdots \end{aligned} \tag{2.54}$$

In his definition of $F(q)$ in (2.51), Ramanujan inadvertently wrote the factor q^2 instead of q on the right-hand side of (2.51). This result is also given in Part II of Sec. 4, where a_n and A_n above are replaced by $\Omega'_2(n)$ and $\Omega_2(n)$, respectively. It is noteworthy that Ramanujan had found a linear combination of modular forms having an Euler product, which is work for which Hecke later became famous [8; 9, pp. 644–707].

3. The Approach of Zhi-Hong Sun and Kenneth Williams Through the Theory of Binary Quadratic Forms

As previously noted in Entries 2.5 and 2.6, on p. 247 of his “lost” notebook, Ramanujan [24] recorded without proof Euler products for the Dirichlet series $\sum_{n=1}^{\infty} a(n)n^{-s}$ and $\sum_{n=1}^{\infty} b(n)n^{-s}$, where the arithmetic functions $a(n)$ and $b(n)$ are defined by

$$\sum_{n=1}^{\infty} a(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{22n}), \quad q \in \mathbb{C}, |q| < 1, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} b(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{3n})(1 - q^{21n}), \quad q \in \mathbb{C}, |q| < 1. \tag{3.2}$$

In [25] Rangachari outlined proofs of Ramanujan’s formulas for the Euler products using class field theory and modular forms. Unfortunately, Ramanujan’s formulas are incorrect, and so Rangachari’s proofs are invalid. The proofs given by Sun and Williams [30] of corrected forms of Ramanujan’s formulas are based on the classical

theory of binary quadratic forms and so are elementary. The corrected forms of Ramanujan's conjectures [30, Theorems 7.2 and 8.2] are

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{1}{1-11^{-s}} \prod_{\substack{p \equiv 2,6,7,8,10 \pmod{11} \\ p \neq 2}} \frac{1}{1-p^{-2s}} \times \prod_{p=3x^2+2xy+4y^2} \frac{1}{1+p^{-s}+p^{-2s}} \prod_{p=x^2+11y^2 \neq 11} \frac{1}{(1-p^{-s})^2} \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \frac{1}{1+7^{-s}} \prod_{\substack{p \equiv 3,5,6 \pmod{7} \\ p \neq 3}} \frac{1}{1-p^{-2s}} \prod_{p \equiv 2,8,11 \pmod{21}} \frac{1}{1+p^{-2s}} \times \prod_{p=x^2+xy+16y^2} \frac{1}{(1-p^{-s})^2} \prod_{p=4x^2+xy+4y^2 \neq 7} \frac{1}{(1+p^{-s})^2}, \tag{3.4}$$

which are valid for $s \in \mathbb{C}$ with $\text{Re } s > 1$. Before describing the approach taken by Sun and Williams in [30], we describe briefly the results we need from the theory of binary quadratic forms.

A binary quadratic form is a polynomial $ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Z}$. We always assume that $ax^2 + bxy + cy^2$ is positive-definite, equivalently $a > 0$ and $b^2 - 4ac < 0$, and primitive, equivalently $\text{gcd}(a, b, c) = 1$. For brevity, we write (a, b, c) for the form $ax^2 + bxy + cy^2$. The discriminant d of the form (a, b, c) is the negative integer $d = b^2 - 4ac$. We note that $d \equiv 0, 1 \pmod{4}$. For $n \in \mathbb{N}$ we define

$$R((a, b, c); n) := \text{card}\{(x, y) \in \mathbb{Z}^2 \mid ax^2 + bxy + cy^2 = n\}$$

so that $R((a, b, c); n)$ counts the number of representations of n by the form (a, b, c) . The class of the form (a, b, c) is the set of forms

$$[a, b, c] := \{a(rx + sy)^2 + b(rx + sy)(tx + uy) + c(tx + uy)^2 \mid r, s, t, u \in \mathbb{Z}, ru - st = 1\}.$$

Each form in $[a, b, c]$ is positive-definite, primitive, and of discriminant d . Clearly the class $[a, b, c]$ contains the form (a, b, c) . Furthermore, if $(A, B, C) \in [a, b, c]$, then $[A, B, C] = [a, b, c]$. Moreover the number of representations of $n \in \mathbb{N}$ by any form in the class $[a, b, c]$ is the same, so we can define

$$R([a, b, c]; n) := R((a, b, c); n).$$

Gauss proved that each positive-definite, primitive, binary quadratic form of discriminant d belongs to one and only one of a finite set $H(d)$ of form classes. We denote the number of such form classes by $h(d)$. With respect to Gaussian composition, which we write multiplicatively, the set $H(d)$ is a finite abelian group. The identity of the group is $I = [1, 0, -d/4]$ if $d \equiv 0 \pmod{4}$ and $I = [1, 1, (1-d)/4]$ if $d \equiv 1 \pmod{4}$. The inverse A^{-1} of the form class $A = [a, b, c]$ is the form class

$[a, -b, c]$. As $H(d)$ is a finite abelian group, there exist $r \in \mathbb{N}$, $A_1, \dots, A_r \in H(d)$, and $h_1, \dots, h_r \in \mathbb{N}$ with $h_1 \cdots h_r = h(d)$, such that

$$H(d) = \{A_1^{k_1} \cdots A_r^{k_r} \mid k_1 = 0, 1, \dots, h_1 - 1, \dots, k_r = 0, 1, \dots, h_r - 1\}.$$

For $n \in \mathbb{N}$ and $M = A_1^{m_1} \cdots A_r^{m_r} \in H(d)$, we define [29, Definition 7.1]

$$F(M, n) := \frac{1}{w(d)} \sum_{k_1=0, \dots, k_r=0}^{h_1-1, \dots, h_r-1} \cos 2\pi \left(\frac{k_1 m_1}{h_1} + \cdots + \frac{k_r m_r}{h_r} \right) R(A_1^{k_1} \cdots A_r^{k_r}; n),$$

where

$$w(d) = \begin{cases} 6, & \text{if } d = -3, \\ 4, & \text{if } d = -4, \\ 2, & \text{if } d < -4. \end{cases}$$

Since each of $H(-3)$ and $H(-4)$ is the trivial group, we have $w(d) = 2$ if $H(d)$ is nontrivial. From [29, Theorem 7.2] we know that $F(M, n)$ is a multiplicative function of $n \in \mathbb{N}$. Sun and Williams [30, p. 372] proved that in $\{s \in \mathbb{C} \mid \operatorname{Re} s > 1\}$ the Dirichlet series $\sum_{n=1}^{\infty} F(M, n)n^{-s}$ converges absolutely, is an analytic function of s , and has an Euler product.

For our purposes we are interested in the function F when $H(d)$ is a cyclic group of order 3 or 4 (so that $w(d) = 2$). If $H(d)$ is a cyclic group of order 3, say, $H(d) = \{I, A, A^2\}$ with $A^3 = I$, $A \neq I$, then

$$F(A, n) = \frac{1}{2}(R(I, n) - R(A, n)) \tag{3.5}$$

is a multiplicative function of n , which is given explicitly in [29, Theorem 10.1]. Similarly, if $H(d)$ is a cyclic group of order 4, say, $H(d) = \{I, A, A^2, A^3\}$ with $A^4 = I$, $A^2 \neq I$, then

$$F(A, n) = \frac{1}{2}(R(I, n) - R(A^2, n)) \tag{3.6}$$

is a multiplicative function of n , whose value is given in [29, Theorem 11.1]. If $H(d)$ is a cyclic group of order ≥ 5 generated by A , then $F(A, n)$ does not have a simple representation such as (3.5) and (3.6); see [29, Theorem 7.4].

We are now in a position to describe the approach taken by Sun and Williams [30]. To keep notation consistent with that of Ramanujan, we replace their function $\phi(q)$ by $f(-q)$, which is defined in (1.2). Next they defined, for $k \in \{1, 2, 3, \dots, 12\}$, the arithmetic function $\phi_k : \mathbb{N} \rightarrow \mathbb{Z}$ by

$$qf(-q^k)f(-q^{24-k}) = \sum_{n=1}^{\infty} \phi_k(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1, \tag{3.7}$$

so that we are interested in $\phi_2(n) = a(n)$ and $\phi_3(n) = b(n)$. Using (1.2) in the left-hand side of (3.7), and manipulating the resulting product of series, Sun and

Williams [30, Theorem 2.2] found, on equating coefficients, explicit formulas for $\phi_k(n)$, $k \in \{1, 2, 3, \dots, 12\}$, namely,

$$\begin{aligned} \phi_1(n) &= \frac{1}{2}(R([1, 1, 6]; n) - R([2, 1, 3]; n)) \quad (d = -23, H(d) \simeq \mathbb{Z}/3\mathbb{Z}), \\ \phi_2(n) &= \frac{1}{2}(R([1, 0, 11]; n) - R([3, 2, 4]; n)) \quad (d = -44, H(d) \simeq \mathbb{Z}/3\mathbb{Z}), \\ \phi_3(n) &= \frac{1}{2}(R([1, 1, 16]; n) - R([4, 1, 4]; n)) \quad (d = -63, H(d) \simeq \mathbb{Z}/4\mathbb{Z}), \\ \phi_4(n) &= \frac{1}{2}(R([1, 0, 20]; n) - R([4, 0, 5]; n)) \quad (d = -80, H(d) \simeq \mathbb{Z}/4\mathbb{Z}), \\ \phi_5(n) &= \frac{1}{2}(R([1, 1, 24]; n) - R([4, 1, 6]; n)) \quad (d = -95, H(d) \simeq \mathbb{Z}/8\mathbb{Z}), \\ \phi_6(n) &= \frac{1}{2}(R([1, 0, 27]; n) - R([4, 2, 7]; n)) \quad (d = -108, H(d) \simeq \mathbb{Z}/3\mathbb{Z}), \\ \phi_7(n) &= \frac{1}{2}(R([1, 1, 30]; n) - R([4, 3, 8]; n)) \quad (d = -119, H(d) \simeq \mathbb{Z}/10\mathbb{Z}), \\ \phi_8(n) &= \frac{1}{2}(R([1, 0, 32]; n) - R([4, 4, 9]; n)) \quad (d = -128, H(d) \simeq \mathbb{Z}/4\mathbb{Z}), \\ \phi_9(n) &= \frac{1}{2}(R([1, 1, 34]; n) - R([4, 3, 9]; n)) \quad (d = -135, H(d) \simeq \mathbb{Z}/6\mathbb{Z}), \\ \phi_{10}(n) &= \frac{1}{2}(R([1, 0, 35]; n) - R([4, 2, 9]; n)) \quad (d = -140, H(d) \simeq \mathbb{Z}/6\mathbb{Z}), \\ \phi_{11}(n) &= \frac{1}{2}(R([1, 1, 36]; n) - R([4, 1, 9]; n)) \quad (d = -143, H(d) \simeq \mathbb{Z}/10\mathbb{Z}), \\ \phi_{12}(n) &= \frac{1}{2}(R([1, 0, 36]; n) - R([4, 0, 9]; n)) \quad (d = -144, H(d) \simeq \mathbb{Z}/4\mathbb{Z}). \end{aligned}$$

In each line of the list above, the value of d is the discriminant of each of the two forms appearing in the formula for $\phi_k(n)$. For $k = 1, 2, \dots, 12$, we have $d = k(k-24)$. The first form class in each line is the identity class of discriminant d . All of the form class groups $H(k(k-24))$, $k = 1, 2, \dots, 12$, are cyclic. Moreover, exactly seven of them have $H(k(k-24)) \simeq \mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

For $k = 1, 2, 6$, we have $d = -23, -44, -108$, respectively, and

$$\begin{aligned} H(-23) &= \{I, A, A^2\}, \quad \text{where } I = [1, 1, 6], A = [2, 1, 3], A^2 = [2, -1, 3], A^3 = I, \\ H(-44) &= \{I, A, A^2\}, \quad \text{where } I = [1, 0, 11], A = [3, 2, 4], A^2 = [3, -2, 4], A^3 = I, \\ H(-108) &= \{I, A, A^2\}, \quad \text{where } I = [1, 0, 27], A = [4, 2, 7], A^2 = [4, -2, 7], A^3 = I. \end{aligned}$$

For these three values of k , we see from the list that

$$\phi_k(n) = \frac{1}{2}(R(I, n) - R(A, n)) = F(A, n). \tag{3.8}$$

Thus $\phi_k(n)$ ($k = 1, 2, 6$) is a multiplicative function of n , whose value is given by [29, Theorem 10.1]; see [30, Theorem 4.4]. Using these evaluations, Sun and Williams [30, Theorem 7.2] deduced that

$$\sum_{n=1}^{\infty} \frac{\phi_1(n)}{n^s} = \frac{1}{1 - 23^{-s}} \prod_{\left(\frac{p}{23}\right)=-1} \frac{1}{1 - p^{-2s}} \times \prod_{p=2x^2+xy+3y^2} \frac{1}{1 + p^{-s} + p^{-2s}} \prod_{p=x^2+xy+6y^2 \neq 23} \frac{1}{(1 - p^{-s})^2}, \tag{3.9}$$

$$\sum_{n=1}^{\infty} \frac{\phi_2(n)}{n^s} = \frac{1}{1 - 11^{-s}} \prod_{\substack{p \equiv 2,6,7,8,10 \pmod{11} \\ p \neq 2}} \frac{1}{1 - p^{-2s}} \times \prod_{p=3x^2+2xy+4y^2} \frac{1}{1 + p^{-s} + p^{-2s}} \prod_{p=x^2+11y^2 \neq 11} \frac{1}{(1 - p^{-s})^2}, \tag{3.10}$$

and

$$\sum_{n=1}^{\infty} \frac{\phi_6(n)}{n^s} = \prod_{p \equiv 5 \pmod{6}} \frac{1}{1 - p^{-2s}} \prod_{p=x^2+27y^2} \frac{1}{(1 - p^{-s})^2} \times \prod_{p=4x^2+2xy+7y^2} \frac{1}{1 + p^{-s} + p^{-2s}}. \tag{3.11}$$

For $k = 3, 4, 8, 12$, we have $d = -63, -80, -128, -144$, respectively, and

$$\begin{aligned} H(-63) &= \{I, A, A^2, A^3\}, \\ &\text{where } I = [1, 1, 16], A = [2, 1, 8], A^2 = [4, 1, 4], A^3 = [2, -1, 8], A^4 = I, \\ H(-80) &= \{I, A, A^2, A^3\}, \\ &\text{where } I = [1, 0, 20], A = [3, 2, 7], A^2 = [4, 0, 5], A^3 = [3, -2, 7], A^4 = I, \\ H(-128) &= \{I, A, A^2, A^3\}, \\ &\text{where } I = [1, 0, 32], A = [3, 2, 11], A^2 = [4, 4, 9], A^3 = [3, -2, 11], A^4 = I, \\ H(-144) &= \{I, A, A^2, A^3\}, \\ &\text{where } I = [1, 0, 36], A = [5, 4, 8], A^2 = [4, 0, 9], A^3 = [5, -4, 8], A^4 = I. \end{aligned} \tag{3.12}$$

For these four values of k , we see from the list that

$$\phi_k(n) = \frac{1}{2}(R(I, n) - R(A^2, n)) = F(A, n).$$

Thus $\phi_k(n)$, $k = 3, 4, 8, 12$, is a multiplicative function of n , whose value is given by [29, Theorem 11.1]; see [30, Theorem 4.5]. Using these evaluations, Sun and

Williams [30, Theorem 8.2] deduced that

$$\sum_{n=1}^{\infty} \frac{\phi_3(n)}{n^s} = \frac{1}{1+7^{-s}} \prod_{\substack{p \equiv 3,5,6 \pmod{7} \\ p \neq 3}} \frac{1}{1-p^{-2s}} \prod_{p \equiv 2,8,11 \pmod{21}} \frac{1}{1+p^{-2s}} \\ \times \prod_{p=x^2+xy+16y^2} \frac{1}{(1-p^{-s})^2} \prod_{p=4x^2+xy+4y^2 \neq 7} \frac{1}{(1+p^{-s})^2}, \quad (3.13)$$

$$\sum_{n=1}^{\infty} \frac{\phi_4(n)}{n^s} = \frac{1}{1+5^{-s}} \prod_{p \equiv 11,13,17,19 \pmod{20}} \frac{1}{1-p^{-2s}} \prod_{p \equiv 3,7 \pmod{20}} \frac{1}{1+p^{-2s}} \\ \times \prod_{p=x^2+20y^2} \frac{1}{(1-p^{-s})^2} \prod_{p=4x^2+5y^2 \neq 5} \frac{1}{(1+p^{-s})^2}, \quad (3.14)$$

$$\sum_{n=1}^{\infty} \frac{\phi_8(n)}{n^s} = \prod_{p \equiv 5,7 \pmod{8}} \frac{1}{1-p^{-2s}} \prod_{p \equiv 3 \pmod{8}} \frac{1}{1+p^{-2s}} \\ \times \prod_{p=x^2+32y^2} \frac{1}{(1-p^{-s})^2} \prod_{p=4x^2+4xy+9y^2 \neq 7} \frac{1}{(1+p^{-s})^2}, \quad (3.15)$$

and

$$\sum_{n=1}^{\infty} \frac{\phi_{12}(n)}{n^s} = \prod_{\substack{p \equiv 3 \pmod{4} \\ p \neq 3}} \frac{1}{1-p^{-2s}} \prod_{p \equiv 5 \pmod{12}} \frac{1}{1+p^{-2s}} \\ \times \prod_{p=x^2+36y^2} \frac{1}{(1-p^{-s})^2} \prod_{p=4x^2+9y^2} \frac{1}{(1+p^{-s})^2}. \quad (3.16)$$

Formulas (3.10) and (3.13) are the corrected formulas (3.3) and (3.4) of Ramanujan. Formulas (3.9), (3.11), (3.14), (3.15), and (3.16) were not stated by Ramanujan.

4. A Partial Manuscript on Euler Products

Pages 233–235 in [24] are devoted to a manuscript by Ramanujan on Euler products in four sections, but in the handwriting of G. N. Watson. The original manuscript can be found in the library of Trinity College, Cambridge. We copy the manuscript section by section and then offer proofs and commentary after each section. As we shall see, Ramanujan discovered many Euler products associated with linear combinations of modular forms. We do not have any ideas on how Ramanujan found these Euler products without invoking the theory of modular forms. It would be extremely interesting to find a new, more elementary method to attack these formulas. In our transcription, we have taken the liberty of introducing standard notation for q -products. Recall also that Ramanujan's Eisenstein series $P(q)$, $Q(q)$, and $R(q)$ are defined by (1.3)–(1.5), respectively.

The following proposition is useful in deducing that $\Omega_1(n) = n\Omega_0(n)$ in the first two series of formulas below.

Proposition 4.1. *Let α be a divisor of 24. Then*

$$q \frac{d}{dq} f_\alpha(z) = f_\alpha(z)P(\alpha z),$$

where $f_\alpha(z) = \eta^{24/\alpha}(\alpha z)$.

Proof. Since

$$\log f_\alpha(z) = \log q + \frac{24}{\alpha} \sum_{n=1}^{\infty} \log(1 - q^{\alpha n}),$$

we arrive at

$$q \frac{d}{dq} f_\alpha(z) = f_\alpha(z) \left(1 - 24 \sum_{n=1}^{\infty} \frac{q^{\alpha n}}{1 - q^{\alpha n}} \right) = f_\alpha(z)P(\alpha z),$$

as desired. □

Another simple proof can be constructed using the theory of modular forms. Observe that $q \frac{d}{dq} f_\alpha(z) - f_\alpha(z)P(\alpha z)$ is a modular form of weight $\frac{12}{\alpha} + 2$ with level $\frac{576}{\alpha^2}$ with a proper quadratic character. Using Sturm’s bound, we only need to check that the first few terms vanish. Note that this method only works for modular forms of integral weight, i.e. when $24/\alpha$ is an even number.

In the following four lists, except for the aforementioned notational simplifications, we quote Ramanujan. In our proofs, we frequently appeal to dimensions of certain spaces of modular forms, all of which can be found in [5] or which can be calculated using MAGMA.

Entry 4.2 (List I). *Suppose that A and B are any two integers such that $A^2 + 3B^2 = p$ and $A \equiv 1 \pmod{3}$, p being a prime of the form $6k + 1$. Let*

$$\sum_{n=1}^{\infty} \Omega_0(n)q^{n/6} = q^{1/6}(q; q)_\infty^4, \tag{4.1}$$

$$\sum_{n=1}^{\infty} \Omega_1(n)q^{n/6} = q^{1/6}(q; q)_\infty^4 P(q), \tag{4.2}$$

$$\sum_{n=1}^{\infty} \Omega_2(n)q^{n/6} = q^{1/6}(q; q)_\infty^4 Q(q),$$

$$\sum_{n=1}^{\infty} \Omega_3(n)q^{n/6} = q^{1/6}(q; q)_\infty^4 R(q), \tag{4.3}$$

$$\sum_{n=1}^{\infty} \Omega'_4(n)q^{n/6} = q^{1/6}(q; q)_\infty^4 Q^2(q), \tag{4.4}$$

$$\sum_{n=1}^{\infty} \Omega_4''(n)q^{n/6} = q^{5/6}(q; q)_{\infty}^{20}, \tag{4.5}$$

$$\Omega_4(n) = \Omega_4'(n) + 288\omega\sqrt{70}\Omega_4''(n), \quad \omega^2 = 1, \tag{4.6}$$

$$\sum_{n=1}^{\infty} \Omega_5(n)q^{n/6} = q^{1/6}(q; q)_{\infty}^4 Q(q)R(q),$$

$$\sum_{n=1}^{\infty} \Omega_7'(n)q^{n/6} = q^{1/6}(q; q)_{\infty}^4 Q^2 R(q), \tag{4.7}$$

$$\sum_{n=1}^{\infty} \Omega_7''(n)q^{n/6} = q^{5/6}(q; q)_{\infty}^{20} R(q), \tag{4.8}$$

$$\Omega_7(n) = \Omega_7'(n) + 10080\omega\sqrt{286}\Omega_7''(n), \quad \omega^2 = 1. \tag{4.9}$$

In all these cases,

$$\sum_{n=1}^{\infty} \frac{\Omega_{\lambda}(n)}{n^s} = \prod_p \frac{1}{1 - \Omega_{\lambda}(p)p^{-s} + p^{2\lambda+1-2s}},$$

where p assumes all prime values greater than 3. If $\lambda = 0, 2, 3, 5$, then

$$\Omega_{\lambda}(p) = \begin{cases} 0, & p \equiv -1 \pmod{6}, \\ (A + iB\sqrt{3})^{2\lambda+1} + (A - iB\sqrt{3})^{2\lambda+1}, & p \equiv 1 \pmod{6}. \end{cases}$$

$\Omega_1(n) = n\Omega_0(n)$ for all values of n . But $\Omega_4(n)$ and $\Omega_7(n)$ do not seem to have such simple laws.

In our arguments below, we always work with forms supported only on integral exponents. This enables us to avoid the use of multiplier systems. Moreover, we note that we proceeded in this fashion throughout Secs. 2 and 3.

Proof of (4.1). For the following facts, we refer to [7, Chap. 3]. The right-hand side of (4.1) equals $\eta^4(6z)$, and this is a modular form of weight 2 and level 36 with trivial character. Though the dimension of this space is 12, its new-space has dimension 1 and its basis element is the unique new form $\eta^4(6z)$. Therefore, its Euler product is given as [12, p. 118]

$$\sum_{n=1}^{\infty} \frac{\Omega_0(n)}{n^s} = \prod_p \frac{1}{1 - \Omega_0(p)p^{-s} + p^{1-2s}}.$$

Now, we give an elementary proof of the explicit formula for $\Omega_0(p)$. First, note that, by the pentagonal number theorem and Jacobi's identity, respectively,

$$\eta(z) = \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{6}}}^{\infty} (-1)^{(n-1)/6} q^{n^2/24},$$

$$\eta^3(z) = \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{4}}}^{\infty} nq^{n^2/8}.$$

Therefore,

$$\Omega_0(p) = \sum_{\substack{n^2+3k^2=4p \\ n \equiv 1 \pmod{6}, k \equiv 1 \pmod{4}}} (-1)^{(n-1)/6} k. \tag{4.10}$$

For integers n and k satisfying the conditions in (4.10), we define

$$\begin{aligned} A &:= \frac{n+3k}{4} \quad \text{and} \quad B := \frac{n-k}{4}, & \text{if } n \equiv 1 \pmod{12}, \\ A &:= \frac{n-3k}{4} \quad \text{and} \quad B := \frac{n+k}{4}, & \text{if } n \equiv 7 \pmod{12}. \end{aligned}$$

Thus, A and B are integers satisfying $A^2 + 3B^2 = p$, with $A \equiv 1 \pmod{3}$. Therefore, from (4.10),

$$\Omega_0(p) = \sum_{\substack{A^2+3B^2=p \\ A \equiv 1 \pmod{3}}} (A - B).$$

Note that if (A, B) satisfies the foregoing conditions, then so does $(A, -B)$. Therefore, we deduce that

$$\Omega_0(p) = \begin{cases} 2A, & \text{if } p \equiv 1 \pmod{6}, p = A^2 + 3B^2, \text{ and } B > 0, \\ 0, & \text{otherwise,} \end{cases}$$

which completes the proof. □

The form $\eta^4(6z)$ is associated with the elliptic curve $y^2 = x^3 + 1$, i.e.

$$\Omega_0(p) = 1 + p - a(p),$$

where $a(p)$ is the number of points on this elliptic curve after reducing modulo p .

Observe that (4.2) follows from Proposition 4.1 and (4.1).

The remaining Euler products with explicit formulas for the p th coefficients can be derived from the fact that these are modular forms with complex multiplication, or, in other words, newforms associated with a certain Hecke Grössencharacter. For the following description, we refer to Ono’s monograph [20].

For the field $K = \mathbb{Q}(\sqrt{-3})$, we can define a Hecke Grössencharacter ϕ by

$$\phi((\alpha)) = \alpha^{k-1},$$

where $k \geq 2$ is an integer, and α is a generator of the ideal (α) , such that $\alpha \equiv 1 \pmod{\Lambda}$, where $\Lambda = (3)$. Then,

$$\Phi(z) := \frac{1}{2} \sum_{\mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})} = \frac{1}{2} \sum_{n=1}^{\infty} a(n) q^n$$

is a newform of weight k of level 36 with a trivial character. Moreover, the ideal (p) is inert if $p \equiv 5 \pmod{6}$, and if $p \equiv 1 \pmod{6}$, then (p) splits in the form

$$(p) = (x + i\sqrt{3}y)(x - i\sqrt{3}y),$$

where x and y are integers such that $x \equiv 1 \pmod{3}$. From these, we deduce that

$$a(p) = \phi((x + i\sqrt{3}y)) + \phi((x - i\sqrt{3}y)) = (x + i\sqrt{3}y)^{k-1} + (x - i\sqrt{3}y)^{k-1},$$

which implies Ramanujan's claim (4.3). Actually (4.3) is identical to Entry 2.7, for which a complete proof was given earlier.

Now we examine the entries that are represented as linear combinations of two forms.

For $\Omega_4(n)$, note that $f_{\Omega_4}(z) = \eta^4(6z)Q^2(6z)$ and $f_{\Omega'_4}(z) = \eta^{20}(6z)$ are in $S_{10}^{\text{new}}(\Gamma_0(36))$, the new space of cusp forms. The dimension of $S_{10}^{\text{new}}(\Gamma_0(36))$ is 4, but there are only two forms for which the exponents are supported on one residue class modulo 6. Note that if $f(z) = q^a \sum a(n)q^{6n} \in S_{10}^{\text{new}}(\Gamma_0(36))$, then a must be coprime to 6.

By a simple calculation for the Hecke operator T_5 , we see that

$$T_5 f_{\Omega_4}(z) = 5806000 f_{\Omega'_4}(z),$$

$$T_5 f_{\Omega'_4}(z) = f_{\Omega_4}(z).$$

The eigenvalues of the matrix

$$\begin{pmatrix} 0 & 5806000 \\ 1 & 0 \end{pmatrix}$$

are $\pm 288\sqrt{70}$. Therefore,

$$f_{\Omega_4}(z) = f_{\Omega'_4}(z) \pm 288\sqrt{70} f_{\Omega_4}(z)$$

is an eigenform for T_5 .

Since T_p and T_5 are commutative, where p is a prime larger than 5, we can conclude that $f_{\Omega_4}(z)$ is a Hecke eigenform as was claimed. Thus, the verifications of (4.4)–(4.6) have been demonstrated.

For $\Omega_7(n)$, the argument is exactly the same as that above. In particular,

$$T_5(\eta^4(6z)Q^2(6z)R(6z)) = 29059430400\eta^{20}(6z)R(6z),$$

$$T_5(\eta^{20}(6z)R(6z)) = \eta^4(6z)Q^2(6z)R(6z).$$

Therefore, the verifications of (4.7)–(4.9) follow.

Entry 4.3 (List II). Suppose that A and B are defined as in List I and let

$$\sum_{n=1}^{\infty} \Omega_0(n)q^{n/3} = q^{1/3}(q; q)_{\infty}^8, \tag{4.11}$$

$$\sum_{n=1}^{\infty} \Omega_1(n)q^{n/3} = q^{1/3}(q; q)_{\infty}^8 P(q), \tag{4.12}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \Omega'_2(n)q^{n/3} &= q^{1/3}(q; q)_{\infty}^8 Q(q), \\ \sum_{n=1}^{\infty} \Omega''_2(n)q^{n/3} &= q^{1/3}(q; q)_{\infty}^{16}, \\ \Omega_2(n) &= \Omega'_2(n) + 6\omega\sqrt{10}\Omega''_2(n), \quad \omega^2 = 1, \\ \sum_{n=1}^{\infty} \Omega_3(n)q^{n/3} &= q^{1/3}(q; q)_{\infty}^8 R(q), \\ \sum_{n=1}^{\infty} \Omega'_4(n)q^{n/3} &= q^{1/3}(q; q)_{\infty}^8 Q^2(q), \\ \sum_{n=1}^{\infty} \Omega''_4(n)q^{n/3} &= q^{2/3}(q; q)_{\infty}^{16} Q(q), \\ \Omega_4(n) &= \Omega'_4(n) + 6\omega\sqrt{70}\Omega''_4(n), \quad \omega^2 = 1, \\ \sum_{n=1}^{\infty} \Omega'_5(n)q^{n/3} &= q^{1/3}(q; q)_{\infty}^8 Q(q)R(q), \\ \sum_{n=1}^{\infty} \Omega''_5(n)q^{n/3} &= q^{2/3}(q; q)_{\infty}^{16} R(q), \\ \Omega_5(n) &= \Omega'_5(n) + 12\omega\sqrt{55}\Omega''_5(n), \quad \omega^2 = 1, \\ \sum_{n=1}^{\infty} \Omega'_7(n)q^{n/3} &= q^{1/3}(q; q)_{\infty}^8 Q^2(q)R(q), \\ \sum_{n=1}^{\infty} \Omega''_7(n)q^{n/3} &= q^{2/3}(q; q)_{\infty}^{16} Q(q)R(q), \\ \Omega_7(n) &= \Omega'_7(n) + 12\omega\sqrt{910}\Omega''_7(n), \quad \omega^2 = 1. \end{aligned}$$

In all these cases,

$$\sum_{n=1}^{\infty} \frac{\Omega_{\lambda}(n)}{n^s} = \prod_p \frac{1}{1 - \Omega_{\lambda}(p)p^{-s} + p^{2\lambda+3-2s}},$$

where p assumes all prime values except 3. If $\lambda = 0$ or 3, then

$$\Omega_{\lambda}(p) = \begin{cases} 0, & \text{if } p \equiv -1 \pmod{3}, \\ (A + iB\sqrt{3})^{2\lambda+3} + (A - iB\sqrt{3})^{2\lambda+3}, & \text{if } p \equiv 1 \pmod{3}. \end{cases}$$

$$\Omega_1(n) = n\Omega_0(n).$$

Note that (4.12) follows from (4.11) and Proposition 4.1. First observe that $\eta^8(3z)$ is a modular form of weight 4 and level 9 with trivial character. Though the dimension of this space is 4, its new-space has dimension 2, and $\eta^8(3z)$ is a basis element and eigenform.

It should be possible to derive explicit formulas by using Hecke Grössen-characters, which we have used in the previous entry. Note that, except for $p = 2$, every prime is congruent to 1 (mod 2). Thus, there is no essential difference between the explicit formulas (4.3) in List I and (4.11) in List II. For the formulas for Ω_4 , Ω_5 , and Ω_7 , the derivations are like those above, and so are omitted.

Here we give another proof for the first entry, (4.11). From [13, p. 373],

$$q(q^3; q^3)^8 = \frac{1}{6} \sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ x \equiv 2 \pmod{3}}} x^3 q^{x^2+3xy+3y^2}.$$

Therefore, for $n \in \mathbb{N}$,

$$\Omega_0(n) = \frac{1}{6} \sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ x \equiv 2 \pmod{3} \\ x^2+3xy+3y^2=n}} x^3.$$

Let n be a prime p . If $p = x^2 + 3xy + 3y^2$, then $p \equiv x^2 \equiv 0, 1 \pmod{3}$. Hence, if $p \equiv 2 \pmod{3}$, then $p \neq x^2 + 3xy + 3y^2$ and so $\Omega_0(p) = 0$. Now suppose that $p \equiv 1 \pmod{3}$. We can define integers A and B uniquely by

$$p = A^2 + 3B^2, \quad A \equiv 1 \pmod{3}, \quad B > 0.$$

We consider the sum

$$\sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ x \equiv 2 \pmod{3} \\ x^2+3xy+3y^2=p}} x^3.$$

If $y = 0$, then $p = x^2$, which is not feasible. If $x + y = 0$, then $y = -x$, so $p = x^2$, which is also not possible. If $x + 2y = 0$, then $x = -2y$, and so $p = y^2$, which leads to a contradiction. Therefore,

$$y \neq 0, \quad x + y \neq 0, \quad x + 2y \neq 0.$$

Moreover, as $p \equiv 1 \pmod{3}$, $(x, y) \not\equiv (0, 0) \pmod{2}$. Thus, we arrive at

$$\begin{aligned} x &= -A - 3B, \quad y = 2B, && \text{if } (x, y) \equiv (1, 0) \pmod{2} \text{ and } y > 0, \\ x &= -A + 3B, \quad y = -2B, && \text{if } (x, y) \equiv (1, 0) \pmod{2} \text{ and } y < 0, \\ x &= -A + 3B, \quad y = A - B, && \text{if } (x, y) \equiv (1, 1) \pmod{2} \text{ and } x + y > 0, \\ x &= -A - 3B, \quad y = -A + B, && \text{if } (x, y) \equiv (1, 1) \pmod{2} \text{ and } x + y < 0, \\ x &= 2A, \quad y = -A + B, && \text{if } (x, y) \equiv (0, 1) \pmod{2} \text{ and } x + 2y > 0, \\ x &= 2A, \quad y = -A - B, && \text{if } (x, y) \equiv (0, 1) \pmod{2} \text{ and } x + 2y < 0. \end{aligned}$$

In summary, for $p \equiv 1 \pmod{3}$,

$$\sum_{\substack{(x,y) \in \mathbb{Z}^2 \\ x \equiv 2 \pmod{3} \\ x^2 + 3xy + 3y^2 = p}} x^3 = 12A^3 - 108AB^2,$$

so $\Omega_0(p) = \frac{1}{6}(12A^3 - 108AB^2) = 2A^3 - 18AB^2$. Finally, we can easily check that $\Omega_0(3) = 0$, which completes the proof.

Entry 4.4 (List III). *Suppose that A and B are integers such that $A^2 + 4B^2 = p$ where p is of the form $4k + 1$.*

$$\sum_{n=1}^{\infty} \Omega_0(n)q^{n/4} = q^{1/4}(q; q)_{\infty}^6, \tag{4.13}$$

$$\sum_{n=1}^{\infty} \Omega_1(n)q^{n/4} = q^{1/4}(q; q)_{\infty}^6 P(q), \tag{4.14}$$

$$\sum_{n=1}^{\infty} \Omega_2(n)q^{n/4} = q^{1/4}(q; q)_{\infty}^6 Q(q),$$

$$\sum_{n=1}^{\infty} \Omega'_3(n)q^{n/4} = q^{1/4}(q; q)_{\infty}^6 R(q),$$

$$\sum_{n=1}^{\infty} \Omega''_3(n)q^{n/4} = q^{3/4}(q; q)_{\infty}^{18},$$

$$\Omega_3(n) = \Omega'_3(n) + 24\omega\sqrt{35}\Omega''_3(n), \quad \omega^2 = -1,$$

$$\sum_{n=1}^{\infty} \Omega_4(n)q^{n/4} = q^{1/4}(q; q)_{\infty}^6 Q^2(q),$$

$$\sum_{n=1}^{\infty} \Omega'_5(n)q^{n/4} = q^{1/4}(q; q)_{\infty}^6 Q(q)R(q),$$

$$\sum_{n=1}^{\infty} \Omega''_5(n)q^{n/4} = q^{3/4}(q; q)_{\infty}^{18} Q(q),$$

$$\Omega_5(n) = \Omega'_5(n) + 24\omega\sqrt{1155}\Omega''_5(n), \quad \omega^2 = -1,$$

$$\sum_{n=1}^{\infty} \Omega'_7(n)q^{n/4} = q^{1/4}(q; q)_{\infty}^6 Q^2(q)R(q),$$

$$\sum_{n=1}^{\infty} \Omega''_7(n)q^{n/4} = q^{3/4}(q; q)_{\infty}^{18} Q^2(q),$$

$$\Omega_7(n) = \Omega'_7(n) + 120\omega\sqrt{3003}\Omega''_7(n), \quad \omega^2 = -1.$$

In all these cases,

$$\sum_{n=1}^{\infty} \frac{\Omega_{\lambda}(n)}{n^s} = \prod_1 \prod_2,$$

where

$$\prod_1 = \prod_p \frac{1}{1 - \Omega_\lambda(p)p^{-s} - p^{2\lambda+2-2s}},$$

p assuming prime values of the form $4k - 1$ and

$$\prod_2 = \prod_p \frac{1}{1 - \Omega_\lambda(p)p^{-s} + p^{2\lambda+2-2s}},$$

p assuming prime values of the form $4k + 1$. If $\lambda = 0, 2,$ or 4 , then

$$\Omega_\lambda(p) = \begin{cases} 0, & p \equiv -1 \pmod{4}, \\ (A + 2iB)^{2\lambda+2} + (A - 2iB)^{2\lambda+2}, & p \equiv 1 \pmod{4}. \end{cases}$$

$$\Omega_1(n) = n\Omega_0(n).$$

As before, (4.14) is a consequence of (4.13) and Proposition 4.1.

Proof of (4.13). First, $\eta^6(4z)$ is again the unique newform in the space $S_3^{\text{new}}(16, (\frac{-4}{\cdot}))$. We derive explicit formulas for each p th coefficient. Let $K = \mathbb{Q}(i)$ and $\Lambda = (2)$. Define a Hecke Grössencharacter by

$$\phi((\alpha)) = \alpha^{k-1},$$

where k is an integer at least equal to 2. Then

$$\Phi(z) := \frac{1}{4} \sum_{\mathfrak{a}} \phi(\mathfrak{a})q^{N(\mathfrak{a})} = \frac{1}{4} \sum_{n=1}^{\infty} a(n)q^n$$

is a newform in the space $S_k^{\text{new}}(16, (\frac{-4}{\cdot}))$. Now $p \equiv -1 \pmod{4}$ is inert in K , and for the primes $p \equiv 1 \pmod{4}$, we have the splitting

$$a(p) = (x + iy)^{k-1} + (x - iy)^{k-1},$$

where x is an odd integer and y is an even integer. (Ramanujan sets $y = 2B$.) This argument also explains the Euler products for Ω_2 and Ω_4 , with $k = 7$ and $k = 11$, respectively.

Now we give an elementary proof of the explicit formula for $\eta^6(4z) = \sum_{n=1}^{\infty} \Omega_0(n)q^n$. Again, we use Jacobi's identity in the form

$$\eta^3(z) = \sum_{\substack{n=1 \\ n \equiv 1 \pmod{4}}}^{\infty} nq^{n^2/8}.$$

Using our previous argument, we arrive at

$$\Omega_0(p) = \sum_{\substack{(C,D) \equiv (1,1) \pmod{4} \\ 2p = C^2 + D^2}} CD. \tag{4.15}$$

Now we define

$$A := \frac{C + D}{2} \quad \text{and} \quad B := \frac{C - D}{2},$$

which imply that A is an odd number and B is an even number. Thus, from (4.15),

$$\Omega_0(p) = \sum_{\substack{A^2+B^2=p \\ (A,B)\equiv(1,0)\pmod{2} \\ (A+B,A-B)\equiv(1,1)\pmod{4}}} A^2 - B^2.$$

Note that if (A, B) satisfies the conditions in the summand, then $(A, -B)$ is the only other pair satisfying the conditions. In summary, we have deduced that

$$\Omega_0(p) = \begin{cases} 2(A^2 - 4B^2), & \text{if } p \equiv 1 \pmod{4} \text{ and } p = A^2 + 4B^2, \\ 0, & \text{if } p \equiv -1 \pmod{4}, \end{cases}$$

which completes the proof. □

For $\Omega_3(n)$, $\Omega_5(n)$, and $\Omega_7(n)$, the derivations are similar to those above.

Now, we give Hecke relations for $\Omega_3(n)$, with $\omega^2 = -1$ for these entries. For the Hecke operator T_3 ,

$$\begin{aligned} T_3(\eta^6(4z)R(q^4)) &= -20160\eta^{18}(4z), \\ T_3(\eta^{18}(4z)) &= \eta^6(4z)R(q^4). \end{aligned}$$

As T_3 and T_p commute for all primes $p > 3$ and the eigenvalues of the matrix

$$\begin{pmatrix} 0 & -20160 \\ 1 & 0 \end{pmatrix}$$

are $\pm 24\sqrt{35}$, we can conclude that

$$f_{\Omega_3}(q) := \eta^6(4z)R(q^4) + \omega 24\sqrt{35}\eta^{18}(4z)$$

is a Hecke eigenform. The other claims can be derived via exactly the same argument, and so we omit the proofs.

Entry 4.5 (List IV). *Let*

$$\sum_{n=1}^{\infty} \Omega_0(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2, \tag{4.16}$$

$$\sum_{n=1}^{\infty} \Omega_1(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2 P(q),$$

$$\sum_{n=1}^{\infty} \Omega'_2(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2 Q(q),$$

$$\sum_{n=1}^{\infty} \Omega''_2(n)q^{n/12} = q^{5/12}(q; q)_{\infty}^{10}, \tag{4.17}$$

$$\Omega_2(n) = \Omega'_2(n) + 48\omega \Omega''_2(n), \quad \omega^2 = 1,$$

$$\sum_{n=1}^{\infty} \Omega'_3(n)q^{n/12} = q^{1/12}(q; q)_{\infty}^2 R(q),$$

$$\begin{aligned}
 \sum_{n=1}^{\infty} \Omega_3''(n)q^{n/12} &= q^{7/12}(q; q)_{\infty}^{14}, \\
 \Omega_3(n) &= \Omega_3'(n) + 360\omega\sqrt{3}\Omega_3''(n), \quad \omega^2 = -1, \\
 \sum_{n=1}^{\infty} \Omega_4'(n)q^{n/12} &= q^{1/12}(q; q)_{\infty}^2 Q^2(q), \\
 \sum_{n=1}^{\infty} \Omega_4''(n)q^{n/12} &= q^{5/12}(q; q)_{\infty}^{10} Q(q), \\
 \Omega_4(n) &= \Omega_4'(n) + 672\omega\Omega_4''(n), \quad \omega^2 = 1, \\
 \sum_{n=1}^{\infty} \Omega_5'(n)q^{n/12} &= q^{1/12}(q; q)_{\infty}^2 Q(q)R(q), \\
 \sum_{n=1}^{\infty} \Omega_5''(n)q^{n/12} &= q^{5/12}(q; q)_{\infty}^{10} R(q), \\
 \sum_{n=1}^{\infty} \Omega_5'''(n)q^{n/12} &= q^{7/12}(q; q)_{\infty}^{14} Q(q), \\
 \sum_{n=1}^{\infty} \Omega_5''''(n)q^{n/12} &= q^{11/12}(q; q)_{\infty}^{22}, \\
 \Omega_5(n) &= \Omega_5'(n) + 96\omega_1\sqrt{1045}\Omega_5''(n) + 216\omega_2\sqrt{7315}\Omega_5'''(n) \\
 &\quad + 103680\omega_1\omega_2\sqrt{7}\Omega_5''''(n), \quad \omega_1^2 = 1, \omega_2^2 = -1, \tag{4.18} \\
 \sum_{n=1}^{\infty} \Omega_7'(n)q^{n/12} &= q^{1/12}(q; q)_{\infty}^2 Q^2(q)R(q), \\
 \sum_{n=1}^{\infty} \Omega_7''(n)q^{n/12} &= q^{5/12}(q; q)_{\infty}^{10} Q(q)R(q), \\
 \sum_{n=1}^{\infty} \Omega_7'''(n)q^{n/12} &= q^{7/12}(q; q)_{\infty}^{14} Q^2(q), \\
 \sum_{n=1}^{\infty} \Omega_7''''(n)q^{n/12} &= q^{11/12}(q; q)_{\infty}^{22} Q(q), \\
 \Omega_7(n) &= \Omega_7'(n) + 48\omega_1\sqrt{910 \cdot 2911}\Omega_7''(n) + 216\omega_2\sqrt{5005 \cdot 2911}\Omega_7'''(n) \\
 &\quad - 471744\omega_1\omega_2\sqrt{22}\Omega_7''''(n), \quad \omega_1^2 = 1, \omega_2^2 = -1.
 \end{aligned}$$

Ramanujan did not provide Euler product formulas for the entries in his final list, Entry 4.5. However, we can provide the missing product formula with

$$\sum_{n=1}^{\infty} \frac{\Omega_{\lambda}(n)}{n^s} = \prod_p \frac{1}{1 - \Omega_{\lambda}(p)p^{-s} + \chi(p)p^{2\lambda-2s}},$$

where χ is the quadratic character modulo 12 defined by

$$\chi(p) = \begin{cases} 1, & \text{if } p \equiv 1, 5 \pmod{12}, \\ -1, & \text{if } p \equiv 7, 11 \pmod{12}. \end{cases} \tag{4.19}$$

These Euler products follow from the general theory of Hecke eigenforms, for example, in [12, Eq. (6.98), p. 118].

First, we give an elementary argument for the evaluation of $a(p)$, where $\eta^2(12z) = \sum_{n=1}^{\infty} a(n)q^n$.

Proof of (4.16). Using the pentagonal number theorem in the form

$$\eta(z) = \sum_{\substack{n=-\infty \\ n \equiv 1 \pmod{6}}}^{\infty} (-1)^{(n-1)/6} q^{n^2/24},$$

we see that

$$a(p) = \sum_{\substack{C^2+D^2=2p \\ (C,D) \equiv (1,1) \pmod{6}}} (-1)^{(C+D-2)/6}.$$

Setting

$$A := \frac{C+D}{2} \quad \text{and} \quad B := \frac{C-D}{2},$$

we observe that $A \equiv 1 \pmod{3}$, $B \equiv 0 \pmod{3}$, and $p = A^2 + B^2$. To satisfy these conditions, p should be congruent to 1 (mod 12), since $p \equiv 1 \pmod{4}$ and $p \equiv 1 \pmod{3}$. On the other hand, when $p \equiv 1 \pmod{12}$ and $p = A^2 + B^2$, then $A \equiv \pm 1 \pmod{3}$ and $B \equiv 0 \pmod{3}$. By employing an argument similar to that used before, we conclude that

$$a(p) = \begin{cases} 2(-1)^{(A-1)/3}, & \text{if } p \equiv 1 \pmod{12} \text{ and } p = A^2 + 9B^2 \text{ with } A \equiv 1 \pmod{3}, \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

Serre [27] proved that every L -series associated to a weight one newform is an Artin L -function attached to an irreducible two-dimensional complex linear representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In the case of $qf^2(-q^{12})$, it is related to the dihedral group, D_4 . Consult [27, pp. 242–244] for further information.

Another Proof of (4.16). We give an elementary proof of (4.16). Throughout the proof, we use the notation from Sec. 3. Recall that from (3.12),

$$H(-144) = \{I, A, A^2, A^3\} = \langle A \rangle \cong \mathbb{Z}/4\mathbb{Z},$$

where

$$I = [1, 0, 36], \quad A = [5, 4, 8], \quad A^2 = [4, 0, 9], \quad A^3 = [5, -4, 8], \quad \text{and} \quad A^4 = I.$$

By [28, Theorem 3.1, p. 16] with $a = b = 1$,

$$(q; q)_\infty^2 = 1 + \sum_{n=1}^\infty \frac{1}{2} (R(I, 12n + 1) - R(A^2, 12n + 1))q^n.$$

As $R(I, 1) = 2$ and $R(A^2, 1) = 0$, we deduce that

$$(q; q)_\infty^2 = \sum_{n=0}^\infty \frac{1}{2} (R(I, 12n + 1) - R(A^2, 12n + 1))q^n. \tag{4.20}$$

With the notation in (3.8),

$$(q; q)_\infty^2 = \sum_{n=0}^\infty \phi_{12}(12n + 1)q^n. \tag{4.21}$$

From (4.20) and (4.21), we deduce that

$$\sum_{n=1}^\infty \Omega_0(n)q^{(n-1)/12} = \sum_{n=0}^\infty \phi_{12}(12n + 1)q^n,$$

so that

$$\Omega_0(n) = \begin{cases} \phi_{12}(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \not\equiv 1 \pmod{12}. \end{cases} \tag{4.22}$$

By [30, Theorem 4.5(iv), p. 371],

$$\phi_{12}(n) = 0, \quad \text{if } n \not\equiv 1 \pmod{12}. \tag{4.23}$$

Thus, from (4.22) and (4.23), we deduce that

$$\Omega_0(n) = \phi_{12}(n), \quad n \in \mathbb{N}. \tag{4.24}$$

By (4.24) and (3.16) (see [30, Theorem 8.2(iv), p. 389]), we obtain

$$\begin{aligned} \sum_{n=1}^\infty \frac{\Omega_0(n)}{n^s} &= \sum_{n=1}^\infty \frac{\phi_{12}(n)}{n^s} = \prod_{\substack{p \equiv 3 \pmod{4} \\ p \neq 3}} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 5 \pmod{12}} \frac{1}{1 + p^{-2s}} \\ &\quad \times \prod_{p=x^2+36y^2} \frac{1}{(1 - p^{-s})^2} \prod_{p=4x^2+9y^2} \frac{1}{(1 + p^{-s})^2}. \end{aligned} \tag{4.25}$$

Let

$$F_s(p) := \frac{1}{1 - \Omega_0(p)p^{-s} + \chi(p)p^{-2s}}. \tag{4.26}$$

By (4.24) and [30, Theorem 4.5(iv), p. 371],

$$\Omega_0(p) = \phi_{12}(p) = \begin{cases} 2, & \text{if } p \equiv 1 \pmod{12}, p = x^2 + 36y^2, \\ -2, & \text{if } p \equiv 1 \pmod{12}, p = 4x^2 + 9y^2, \\ 0, & \text{if } p \not\equiv 1 \pmod{12}. \end{cases} \tag{4.27}$$

From the definition of $\chi(p)$ in (4.19), (4.26), and (4.27), we deduce that

$$F_s(2) = 1, \tag{4.28}$$

$$F_s(3) = 1, \tag{4.29}$$

$$F_s(p) = \begin{cases} \frac{1}{1 - 2p^{-s} + p^{-2s}} = \frac{1}{(1 - p^{-s})^2}, & \text{if } p \equiv 1 \pmod{12}, p = x^2 + 36y^2, \\ \frac{1}{1 + 2p^{-s} + p^{-2s}} = \frac{1}{(1 + p^{-s})^2}, & \text{if } p \equiv 1 \pmod{12}, p = 4x^2 + 9y^2, \\ \frac{1}{1 + p^{-s}}, & \text{if } p \equiv 5 \pmod{12}, \\ \frac{1}{1 - p^{-2s}}, & \text{if } p \equiv 3 \pmod{4}, p \neq 3. \end{cases} \tag{4.30}$$

Thus, appealing to (4.25), (4.28)–(4.30), and (4.26), we obtain

$$\sum_{n=1}^{\infty} \frac{\Omega_0(n)}{n^s} = \prod_p F_s(p) = \prod_p \frac{1}{1 - \Omega_0(p)p^{-s} + \chi(p)p^{-2s}}$$

as asserted.

We have shown that Ramanujan’s missing product formula for $\lambda = 0$ in Entry 4.5 is the formula (3.16) of Sun and Williams. By [29, Theorem 7.4(ii)], $\phi_{12}(n)$ is a multiplicative function of n , so by (4.24), $\Omega_0(n)$ is a multiplicative function of n . □

For Ω_2, Ω_3 , and Ω_4 , we can verify Ramanujan’s claims by using the same argument, so we omit the proofs here. For $\Omega_5(n)$, we give a more detailed verification. Note that $f_1(q) := \eta^2(12z)Q(q^{12})R(q^{12})$, $f_5(q) := \eta^{10}(12z)R(q^{12})$, $f_7(q) := \eta^{14}(12z)Q(q^{12})$, and $f_{11}(q) := \eta^{22}(12z)$ are in the space $S_{11}^{\text{new}}(\Gamma_0(144), \chi)$, where χ is defined in (4.19). Moreover, each form f_a is supported on one residue class $a \pmod{12}$, which is coprime to 12. We can easily observe that $T_p f_a$ is supported on the residue class $pa \pmod{12}$ and that

$$T_5 \begin{pmatrix} f_1 \\ f_5 \\ f_7 \\ f_{11} \end{pmatrix} = \begin{pmatrix} 0 & 963027 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 46080 \\ 0 & 0 & 209 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_5 \\ f_7 \\ f_{11} \end{pmatrix}.$$

Thus, we find that

$$f_1 + \omega_1 96\sqrt{1045}f_5 \quad \text{and} \quad f_7 + \omega_1 \frac{96}{209}\sqrt{1045}f_{11}$$

are eigenforms with eigenvalue $96^2 \cdot 1045$ under the action of T_5 . By a similar calculation, we observe that

$$f_1 + \omega_2 216\sqrt{7315}f_7 \quad \text{and} \quad f_5 + \omega_2 \frac{216}{209}\sqrt{7315}f_{11}$$

are eigenforms under the action of T_7 , and

$$f_1 + \omega_3 103680\sqrt{7}f_{11} \quad \text{and} \quad f_5 + \omega_3 \frac{103680}{46080}\sqrt{7}f_7$$

are eigenforms under the action of T_{11} , where $\omega_3^2 = -1$. Therefore,

$$\begin{aligned} & f_1 + \omega_1 96\sqrt{1045}f_5 + \omega_2 216\sqrt{7 \cdot 1045} \left(f_7 + \omega_1 \frac{96}{209}\sqrt{1045}f_{11} \right) \\ &= f_1 + \omega_2 216\sqrt{7315}f_7 + \omega_1 96\sqrt{1045} \left(f_5 + \omega_2 \frac{216}{209}\sqrt{7 \cdot 1045}f_{11} \right) \\ &= f_1 + \omega_3 193680\sqrt{7}f_{11} + \omega_1 96\sqrt{1045} \left(f_5 + \omega_3 \frac{103680}{46080}\sqrt{7}f_7 \right) \\ &= f_1 + \omega_1 96\sqrt{1045}f_5 + \omega_2 216\sqrt{7 \cdot 1045} + \omega_1 \omega_2 103680\sqrt{7}f_{11} \end{aligned}$$

is the Hecke eigenform as desired. For Ω_7 , we can use exactly the same argument, so we omit it. We remark that Rangachari [25] pointed out that the coefficient in the definition of $\Omega_7(n)$ should be 4717440 instead of 471744, as was written by Ramanujan.

We further remark that an approach of Chan, Cooper and Toh [4] can be used to derive representations for certain coefficients that Ramanujan did not provide. For example, Theorem 7.1 in [4] implies that

$$\sum_{n=1}^{\infty} \Omega'_2(n)q^{n/12} = \eta^2(z)Q(q) = \frac{1}{4} \sum_{\substack{\alpha \equiv 1 \pmod{6} \\ \beta \equiv 1 \pmod{6}}} (-1)^{(\alpha+\beta+4)/6} (\alpha + i\beta)^4 q^{(\alpha^2+\beta^2)/24},$$

where $\Omega'_2(n)$ is defined in List IV, and Theorem 7.4 in [4] implies that

$$\sum_{n=1}^{\infty} \Omega_2(n)q^{n/4} = \eta^6(z)Q(q) = -\frac{1}{8i} \sum_{\substack{\alpha \equiv 1 \pmod{4} \\ \beta \equiv 1 \pmod{4}}} (\alpha + i\beta)^6 q^{(\alpha^2+\beta^2)/12},$$

which gives another explicit formula for $\Omega_2(n)$ in List III.

The results in [4, Sec. 7] can also be used to establish some of the formulas in Lists I–III that we derived by employing Hecke Grössencharacters.

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References

- [1] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part III* (Springer, New York, 2012).
- [2] B. C. Berndt and K. Ono, Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary, *Sém. Lothar. Combin.* **42** (1999), Art. B42c, 63 pp.; also in *The Andrews Festschrift*, eds. D. Foata and G.-N. Han (Springer-Verlag, Berlin, 2001), pp. 39–110.
- [3] H. H. Chan, S. Cooper and W.-C. Liaw, On $\eta^3(a\tau)\eta^3(b\tau)$ with $a + b = 8$, *J. Austral. Math. Soc.* **84** (2008) 301–313.
- [4] H. H. Chan, S. Cooper and P. C. Toh, Ramanujan's Eisenstein series and powers of Dedekind's eta-function, *J. London Math. Soc. (2)* **75** (2007) 225–242.
- [5] H. Cohen and J. Oesterlé, Dimensions des espaces de formes modulaires, in *Modular Functions of One Variable VI*, Lecture Notes in Mathematics, Vol. 627 (Springer-Verlag, Berlin, 1977), pp. 69–78.
- [6] P. Deligne, La conjecture de Weil. I, *Publ. Math. Inst. Hautes Études Sci.* **43** (1974) 273–307.
- [7] F. Diamond and J. Shurman, *A First Course in Modular Forms* (Springer, New York, 2005).
- [8] E. Hecke, Über Modulfunctionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung, Part I, *Math. Ann.* **114** (1937) 1–28; Part II, *Math. Ann.* **114** (1937) 316–351.
- [9] ———, *Mathematische Werke* (Vandenhoeck & Ruprecht, Göttingen, 1970).
- [10] L. K. Hua, *Introduction to Number Theory* (Springer-Verlag, Berlin, 1982).
- [11] J. G. Huard, P. Kaplan and K. S. Williams, The Chowla–Selberg formula for genera, *Acta Arith.* **73** (1995) 271–301.
- [12] H. Iwaniec, *Topics in Classical Automorphic Forms* (American Mathematical Society, Providence, RI, 1997).
- [13] F. Klein and R. Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunctionen, Band 2* (Teubner, Leipzig, 1892).
- [14] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms* (Springer-Verlag, New York, 1984).
- [15] G. Köhler, *Eta Products and Theta Series Identities* (Springer, Berlin, 2011).
- [16] Y. Martin, Multiplicative η -products, *Trans. Amer. Math. Soc.* **348** (1996) 4825–4856.
- [17] L. J. Mordell, On Mr. Ramanujan's empirical expansions of modular functions, *Proc. Cambridge Philos. Soc.* **19** (1917) 117–124.
- [18] M. Newman, Modular forms whose coefficients possess multiplicative properties, *Ann. of Math.* **70** (1959) 478–489.
- [19] ———, Construction and application of a class of modular functions II, *Proc. London Math. Soc. (3)* **9** (1959) 373–387.
- [20] K. Ono, *The Web of Modularity* (American Mathematical Society, Providence, RI, 2004).
- [21] S. Raghavan, On Ramanujan and Dirichlet series with Euler products, *Glasgow Math. J.* **25** (1984) 203–206.
- [22] S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Philos. Soc.* **22** (1916) 159–184.
- [23] ———, *Collected Papers* (Cambridge University Press, Cambridge, 1927); reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.
- [24] ———, *The Lost Notebook and Other Unpublished Papers* (Narosa, New Delhi, 1988).

- [25] S. S. Rangachari, Ramanujan and Dirichlet series with Euler products, *Proc. Indian Acad. Sci. (Math. Sci.)* **91** (1982) 1–15.
- [26] ———, Euler products, modular identities and elliptic integrals in Ramanujan's manuscripts, II, in *Ramanujan Revisited*, eds. G. E. Andrews, R. A. Askey, B. C. Berndt, K. G. Ramanathan and R. A. Rankin (Academic Press, Boston, 1988), pp. 347–357.
- [27] J.-P. Serre, Modular forms of weight one and Galois representations, in *Algebraic Number Fields: L-Functions and Galois Properties* (Academic Press, London, 1977), pp. 193–268.
- [28] Z.-H. Sun, The expansion of $\prod_{k=1}^{\infty} (1 - q^{ak})(1 - q^{bk})$, *Acta Arith.* **134** (2008) 11–29.
- [29] Z.-H. Sun and K. S. Williams, On the number of representations of n by $ax^2 + bxy + cy^2$, *Acta Arith.* **122** (2006) 101–171.
- [30] ———, Ramanujan identities and Euler products for a type of Dirichlet series, *Acta Arith.* **122** (2006) 349–393.