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THE PROPORTION OF CYCLIC QUARTIC FIELD DISCRIMINANTS DIVISIBLE BY A GIVEN PRIME

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Abstract

Let $x \in \mathbb{R}$ and q a fixed prime. Let $S(x;q)$ be the set of positive integers $n \leq x$ with n equal to the discriminant of some cyclic quartic field and n divisible by q. An asymptotic formula is given for $\operatorname{card} S(x;q)$ as $x \to +\infty$.

1. Introduction

Let $x \in \mathbb{R}$. Let $S(x)$ denote the set of positive integers $n \leq x$ with n equal to the discriminant of some cyclic quartic field. Let $C(x) = \operatorname{card} S(x)$. It was shown in [2] that

$$C(x) = \frac{11}{2\pi^2}\left(\frac{88+\sqrt{2}}{88}C - 1\right)x^{1/2} + O(x^{1/3}\log x), \tag{1.1}$$

as $x \to +\infty$, where

$$C = \prod_{p \equiv 1 \pmod 4}\left(1 + \frac{1}{(p+1)\sqrt{p}}\right). \tag{1.2}$$

Here and throughout this paper p denotes a prime.

Let q be a fixed prime. Let $S(x;q)$ be the set of positive integers $n \leq x$ with n equal to the discriminant of some cyclic quartic field and n divisible by q. We determine an asymptotic formula for the number $C_q(x) = \operatorname{card} S(x;q)$ of cyclic quartic field discriminants which are $\leq x$ and divisible by the prime q, as $x \to +\infty$. We prove

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THEOREM. *Let q be a prime. Then*

$$C_q(x) = E_q x^{1/2} + O\big(x^{1/3}\log x\big), \tag{1.3}$$

as $x \to +\infty$, where

$$E_2 = \frac{3}{2\pi^2}\left(\frac{24+\sqrt{2}}{24}C - 1\right),$$

$$E_q = \frac{11}{2\pi^2(q+1)}\left(\frac{88+\sqrt{2}}{88}C - 1\right), \qquad \textit{if} \quad q \equiv 3 \pmod 4,$$

and

$$E_q = \frac{11}{2\pi^2(q+1)}\left(\frac{88+\sqrt{2}}{88}\lambda(q)C - 1\right), \qquad \textit{if} \quad q \equiv 1 \pmod 4,$$

where

$$\lambda(q) := \frac{1+\frac{1}{\sqrt{q}}}{1+\frac{1}{(q+1)\sqrt{q}}}.$$

The proportion of cyclic quartic field discriminants divisible by the fixed prime q is

$$d_q = \lim_{x\to+\infty}\frac{C_q(x)}{C(x)} = \frac{E_q}{\frac{11}{2\pi^2}\left\{\frac{88+\sqrt{2}}{88}C - 1\right\}}.$$

Appealing to the values of E_q given in the theorem, the proportion d_q is given by

$$d_q = \frac{(24+\sqrt{2})C - 24}{(88+\sqrt{2})C - 88}, \qquad \text{if} \quad q = 2,$$

$$d_q = \frac{1}{q+1}, \qquad \text{if} \quad q \equiv 3 \pmod 4,$$

$$d_q = \frac{(88+\sqrt{2})\lambda(q)C - 88}{(q+1)((88+\sqrt{2})C - 88)}, \qquad \text{if} \quad q \equiv 1 \pmod 4.$$

2. Notation and Lemmas

We make considerable use of the notation and results in [1], [2] and [3]. As in [1, eqn. (3.7), p. 100] we set

$$\wp = \{n \in \mathbb{N} \mid n = p_1p_2\cdots p_m \quad (m \ge 1),$$
$$p_1,\ldots,p_m \text{ (distinct primes)} \equiv 1 \pmod 4\}.$$

Analogous to the sums $S_1(x)$ and $S_2(x)$ defined in [3, p. 142], we define the sums $T_1(x)$ and $T_2(x)$ as follows.

DEFINITION 2.1. For a prime q we set

$$T_1(x) = \sum_{\substack{D \le x^{1/3} \\ D \in \wp \\ q \nmid D}} \sum_{\substack{1 \le A \le \sqrt{xD^{-3}} \\ (A,2D)=1 \\ A \text{ sqf} \\ q \mid A}} 1,$$

and for a prime $q \equiv 1 \pmod 4$ we set

$$T_2(x) = \sum_{\substack{D \le x^{1/3} \\ D \in \wp \\ q \mid D}} \sum_{\substack{1 \le A \le \sqrt{xD^{-3}} \\ (A,2D)=1 \\ A \text{ sqf} \\ q \nmid A}} 1.$$

Here and throughout this paper "sqf" indicates squarefree. It is also convenient to define a constant C', which is closely related to C.

DEFINITION 2.2. Let q be a prime. Set

$$C' = \prod_{\substack{p \equiv 1 \pmod 4 \\ p \ne q}} \left(1 + \frac{1}{(p+1)\sqrt{p}}\right) = \begin{cases} C, & \text{if } q \not\equiv 1 \pmod 4, \\ \frac{C}{1+\frac{1}{(q+1)\sqrt{q}}}, & \text{if } q \equiv 1 \pmod 4. \end{cases}$$

We begin by proving an asymptotic formula for $T_1(x)$ as $x \to +\infty$.

LEMMA 2.1. *Let q be a prime. Then*

$$T_1(x) = \frac{4}{\pi^2} \frac{x^{1/2}}{q+1}(C'-1) + O\big(x^{1/3}\log x\big),$$

as $x \to +\infty$, where the constant implied by the O-symbol is absolute.

PROOF. In [3, Lemma 2.2] an asymptotic formula is given for the sum

$$\sum_{\substack{1 \le n \le x \\ (n,k)=1 \\ n \text{ sqf} \\ q \mid n}} 1.$$

Appealing to this formula, we obtain

$$\begin{aligned} T_1(x) &= \sum_{\substack{D \le x^{1/3} \\ D \in \wp \\ q \nmid D}} \frac{x^{1/2}}{D^{3/2}} \frac{1}{q+1} \frac{6}{\pi^2} \frac{\phi(2D)}{2D} \prod_{p \mid 2D} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &\qquad + O\left(x^{1/4} q^{-1/2} \sum_{D \le x^{1/3}} d(D) D^{-3/4} \right) \\ &= \frac{4}{\pi^2} \frac{x^{1/2}}{q+1} \sum_{\substack{D \le x^{1/3} \\ D \in \wp \\ q \nmid D}} D^{-5/2} \phi(D) \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &\qquad + O\left(x^{1/4} q^{-1/2} \sum_{D \le x^{1/3}} d(D) D^{-3/4} \right). \end{aligned}$$

In [2, Lemma 6] it is shown that

$$\sum_{1 \le n \le x} \frac{d(n)}{n^{3/4}} = O\big(x^{1/4} \log x\big).$$

Appealing to this result, we see that

$$\sum_{D \le x^{1/3}} d(D) D^{-3/4} = O\big(x^{1/12} \log x\big). \tag{2.1}$$

Also

$$\begin{aligned} \sum_{\substack{D \le x^{1/3} \\ D \in \wp \\ q \nmid D}} D^{-5/2} \phi(D) \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} &= \sum_{\substack{D=1 \\ D \in \wp \\ q \nmid D}}^{\infty} D^{-5/2} \phi(D) \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &\qquad + O\left(\sum_{D > x^{1/3}} D^{-5/2} \phi(D) \right), \end{aligned}$$

as

$$1 < \prod_{p \mid D} \left(1 - \frac{1}{p^2}\right)^{-1} < \frac{\pi^2}{6}.$$

Clearly

$$\sum_{\substack{D=1\\ D\in\wp\\ q\nmid D}}^{\infty} D^{-5/2}\phi(D)\prod_{p|D}\left(1-\frac{1}{p^2}\right)^{-1} = \prod_{\substack{p\equiv 1 \pmod 4\\ p\neq q}}\left(1+\frac{1}{(p+1)\sqrt{p}}\right)-1 = C'-1.$$

Also

$$\sum_{D>x^{1/3}} D^{-5/2}\phi(D) = O\left(\sum_{D>x^{1/3}} D^{-3/2}\right) = O\big(x^{-1/6}\big).$$

Putting these results together, we obtain

$$T_1(x) = \frac{4}{\pi^2}\frac{x^{1/2}}{q+1}(C'-1) + O\big(x^{1/3}\log x\big),$$

as $x \to +\infty$, as asserted.

LEMMA 2.2. *Let q be a prime $\equiv 1 \pmod 4$. Then*

$$T_2(x) = \frac{4}{\pi^2}\frac{x^{1/2}}{(q+1)\sqrt{q}}C' + O\big(x^{1/3}\log x\big),$$

as $x \to +\infty$, where the constant implied by the O-symbol is absolute.

PROOF. In [3, Lemma 2.1, p. 142] an asymptotic formula for

$$\sum_{\substack{1\le n\le x\\ n \text{ sqf}\\ (n,k)=1}} 1$$

is given. Using this formula we obtain

$$T_2(x) = \sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q|D}}\left(\frac{x^{1/2}}{D^{3/2}}\frac{6}{\pi^2}\frac{\phi(2D)}{2D}\prod_{p|2D}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(\left(\frac{x}{D^3}\right)^{1/4} d(D)\right)\right)$$

$$= \frac{4}{\pi^2}x^{1/2}\sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q|D}} D^{-5/2}\phi(D)\prod_{p|D}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(x^{1/4}\sum_{D\le x^{1/3}} d(D)D^{-3/4}\right).$$

From (2.1) we see that

$$x^{1/4}\sum_{D\le x^{1/3}} d(D)D^{-3/4} = O\big(x^{1/4}x^{1/12}\log x\big) = O\big(x^{1/3}\log x\big).$$

Also

$$\sum_{\substack{D\le x^{1/3}\\ D\in\wp\\ q|D}} D^{-5/2}\phi(D)\prod_{p|D}\left(1-\frac{1}{p^2}\right)^{-1}$$

$$= \sum_{\substack{D=1\\ D\in\wp\\ q|D}}^{\infty} D^{-5/2}\phi(D)\prod_{p|D}\left(1-\frac{1}{p^2}\right)^{-1} + O\left(\sum_{D>x^{1/3}} D^{-5/2}\phi(D)\right)$$

$$= q^{-5/2}\phi(q)\left(1-\frac{1}{q^2}\right)^{-1}\prod_{\substack{p\equiv 1 \pmod 4\\ p\neq q}}\left(1+\frac{1}{(p+1)\sqrt{p}}\right) + O(x^{-1/6})$$

$$= \frac{1}{(q+1)\sqrt{q}}C' + O(x^{-1/6}).$$

as

$$\sum_{D>x^{1/3}} D^{-5/2}\phi(D) = O\left(\sum_{D>x^{1/3}} D^{-3/2}\right) = O(x^{-1/6}).$$

Thus

$$\begin{aligned} T_2(x) &= \frac{4}{\pi^2}\frac{x^{1/2}}{(q+1)\sqrt{q}}C' + O(x^{1/2-1/6}) + O(x^{1/3}\log x) \\ &= \frac{4}{\pi^2}\frac{x^{1/2}}{(q+1)\sqrt{q}}C' + O(x^{1/3}\log x), \end{aligned}$$

as asserted.

3. Proof of Theorem

We recall from [2, Theorem 4, p. 191] the following result.

PROPOSITION. *Let $n\in\mathbb{N}$. Then*

$$n = d(K) \quad \textit{for some cyclic quartic field} \quad K$$

if and only if

$n = A^2D^3, 2^4A^2D^3, 2^6A^2D^3$ *or* $2^{11}A^2D^3$
for some $D\in\wp$ *and some odd positive squarefree integer* A
coprime with D

or

$$n = 2^{11}A^2 \quad \textit{for some odd positive squarefree integer} \quad A.$$

First we consider the case $q = 2$. From the proposition just quoted we see that

$$C_2(x) = \sum_{\alpha\in\{2,3,11/2\}} \sum_{\substack{D\leq x^{1/3} \\ D\in\wp}} \sum_{\substack{1\leq A\leq (x/2^{2\alpha}D^3)^{1/2} \\ (A,2D)=1 \\ A \text{ sqf}}} 1 + \sum_{\substack{1\leq A\leq (x/2^{11})^{1/2} \\ A \text{ odd} \\ A \text{ sqf}}} 1$$

so that

$$C_2(x) = \sum_{\alpha\in\{2,3,11/2\}} \sum_{\substack{D\leq x^{1/3} \\ D\in\wp}} T\big(x^{1/2}D^{-3/2}2^{-\alpha}\big) + E\big(x^{1/2}2^{-11/2}\big),$$

where

$$T(x) = \sum_{\substack{1\leq A\leq x \\ (A,2D)=1 \\ A \text{ sqf}}} 1, \quad E(x) = \sum_{\substack{1\leq A\leq x \\ A \text{ odd} \\ A \text{ sqf}}} 1.$$

From [2, eqn. (5.8), p. 192] we have

$$E\big(x^{1/2}2^{-11/2}\big) = \frac{1}{2^{7/2}\pi^2}x^{1/2} + O\big(x^{1/4}\big).$$

By exactly the same argument as in [2, pp. 192–193] with $\alpha = 0$ omitted, we obtain

$$\sum_{\alpha\in\{2,3,11/2\}} \sum_{\substack{D\leq x^{1/3} \\ D\in\wp}} T\big(x^{1/2}D^{-3/2}2^{-\alpha}\big) = \sum_{\alpha\in\{2,3,11/2\}} \frac{4x^{1/2}}{2^\alpha\pi^2}(C-1) + O\big(x^{1/3}\log x\big).$$

Hence

$$\begin{aligned} C_2(x) &= \left(\frac{x^{1/2}}{\pi^2} + \frac{x^{1/2}}{2\pi^2} + \frac{x^{1/2}}{2^{7/2}\pi^2}\right)(C-1) + O\big(x^{1/3}\log x\big) + \frac{x^{1/2}}{2^{7/2}\pi^2} \\ &= \frac{3}{2\pi^2}\left(\frac{24+\sqrt{2}}{24}C - 1\right)x^{1/2} + O\big(x^{1/3}\log x\big), \end{aligned}$$

as asserted.

Next we find an asymptotic formula for $C_q(x)$ when q is a prime with $q \equiv 3 \pmod 4$. From the proposition (remembering that $D \in \wp$ cannot be divisible by q), we see that for $q \equiv 3 \pmod 4$ we have

$$C_q(x) = \sum_{\alpha\in\{0,4,6,11\}} \sum_{\substack{D\leq x^{1/3}\\ D\in\wp}} \sum_{\substack{1\leq A\leq (x/2^\alpha D^3)^{1/2}\\ (A,2D)=1\\ A \text{ sqf}\\ q\mid A}} 1 + \sum_{\substack{1\leq A\leq (x/2^{11})^{1/2}\\ A \text{ odd}\\ A \text{ sqf}\\ q\mid A}} 1$$

so that

$$C_q(x) = \sum_{\alpha\in\{0,4,6,11\}} T_1(2^{-\alpha}x) + \sum_{\substack{1\leq A\leq (x/2^{11})^{1/2}\\ A \text{ odd}\\ A \text{ sqf}\\ q\mid A}} 1. \tag{3.1}$$

Appealing to [3, Lemma 2.2, p. 142] the second sum on the right hand side of (3.1) is

$$\left(\frac{x}{2^{11}}\right)^{1/2} \frac{1}{q+1} \frac{6}{\pi^2} \frac{\phi(2)}{2} \prod_{p\mid 2}\left(1-\frac{1}{p^2}\right)^{-1} + O(x^{1/4}) = \frac{x^{1/2}}{(q+1)\pi^2} 2^{-7/2} + O(x^{1/4}).$$

Then, appealing to Lemma 2.1, we obtain

$$\begin{aligned} C_q(x) &= \sum_{\alpha\in\{0,4,6,11\}} T_1(2^{-\alpha}x) + \frac{2^{-7/2}}{(q+1)\pi^2} x^{1/2} + O(x^{1/4}) \\ &= \frac{(C'-1)}{(q+1)\pi^2} x^{1/2} \left(\frac{4}{2^{11/2}} + \frac{4}{2^3} + \frac{4}{2^2} + 4\right) + O(x^{1/3}\log x) \\ &\quad + \frac{2^{-7/2}}{(q+1)\pi^2} x^{1/2} + O(x^{1/4}) \\ &= \frac{11}{2\pi^2(q+1)} \left(\frac{88+\sqrt{2}}{88} C - 1\right) x^{1/2} + O(x^{1/3}\log x), \end{aligned}$$

as $C' = C$ when $q \equiv 3 \pmod 4$.

Finally we determine an asymptotic formula for $C_q(x)$ when q is a prime with $q \equiv 1 \pmod 4$. From the proposition we obtain for $q \equiv 1 \pmod 4$

$$C_q(x) = \sum_{\alpha\in\{0,4,6,11\}} \sum_{\substack{D\leq x^{1/3}\\ D\in\wp\\ q\mid D}} \sum_{\substack{1\leq A\leq (x/2^\alpha D^3)^{1/2}\\ (A,2D)=1\\ A \text{ sqf}\\ q\nmid A}} 1$$

$$+ \sum_{\alpha \in \{0,4,6,11\}} \sum_{\substack{D \le x^{1/3} \\ D \in \wp \\ q \nmid D}} \sum_{\substack{1 \le A \le (x/2^{\alpha} D^3)^{1/2} \\ (A,2D)=1 \\ A \text{ sqf} \\ q \mid A}} 1$$

$$+ \sum_{\substack{1 \le A \le (x/2^{11})^{1/2} \\ A \text{ odd} \\ A \text{ sqf} \\ q \mid A}} 1$$

so that

$$C_q(x) = \sum_{\alpha \in \{0,4,6,11\}} T_1(2^{-\alpha} x) + \sum_{\alpha \in \{0,4,6,11\}} T_2(2^{-\alpha} x) + \sum_{\substack{1 \le A \le (x/2^{11})^{1/2} \\ A \text{ odd} \\ A \text{ sqf} \\ q \mid A}} 1. \tag{3.2}$$

Exactly as in the determination of $C_q(x)$ when $q \equiv 3 \pmod 4$, we obtain

$$\begin{aligned} &\sum_{\alpha \in \{0,4,6,11\}} T_1(2^{-\alpha} x) + \sum_{\substack{A \le (x/2^{11})^{1/2} \\ A \text{ odd} \\ A \text{ sqf} \\ q \mid A}} 1 \\ &= \frac{11}{2\pi^2 (q+1)} \left(\frac{88+\sqrt{2}}{88} C' - 1 \right) x^{1/2} + O\big(x^{1/3} \log x\big). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), and appealing to Lemma 2.2, we obtain

$$\begin{aligned} C_q(x) &= \frac{11}{2\pi^2 (q+1)} \left(\frac{88+\sqrt{2}}{88} C' - 1 \right) x^{1/2} \\ &\quad + \frac{4}{\pi^2} \frac{x^{1/2}}{(q+1)\sqrt{q}} C' \big(1 + 2^{-11/2} + 2^{-2} + 2^{-3}\big) + O\big(x^{1/3} \log x\big) \\ &= \frac{11}{2\pi^2 (q+1)} \left(\frac{88+\sqrt{2}}{88} \left(1 + \frac{1}{\sqrt{q}}\right) C' - 1 \right) x^{1/2} + O\big(x^{1/3} \log x\big) \\ &= \frac{11}{2\pi^2 (q+1)} \left(\frac{88+\sqrt{2}}{88} \lambda(q) C - 1 \right) x^{1/2} + O\big(x^{1/3} \log x\big), \end{aligned}$$

as

$$\left(1 + \frac{1}{\sqrt{q}}\right) C' = \lambda(q) C,$$

when $q \equiv 1 \pmod 4$. This completes the proof.

References

[1] Z.M. Ou and K.S. Williams, On the density of cyclic quartic fields, Canad. Math. Bull. **44** (2001), 97–104.

[2] B.K. Spearman and K.S. Williams, Integers which are discriminants of bicyclic or cyclic quartic fields, JP J. Algebra Number Theory Appl. **1** (2001), 179–194.

[3] B.K. Spearman and K.S. Williams, The proportion of cyclic quartic fields with discriminant divisible by a given prime, Proc. Japan Acad. Ser. A. **80** (2004), 141–145.

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