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Values of the Euler phi function not divisible by a given odd prime

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Abstract. An asymptotic formula is given for the number of integers $n \leq x$ for which $\phi(n)$ is not divisible by a given odd prime.

1. Introduction

We denote the set of natural numbers by $\mathbf{N}$ and the set of integers by $\mathbf{Z}$. If $a \in \mathbf{Z}$ and $b \in \mathbf{Z}$ are not both 0, we denote the greatest common divisor of a and b by (a, b). We let ϕ denote Euler's phi function so that for $n \in \mathbf{N}$ we have

$$(1) \qquad \phi(n) := \operatorname{card}\{m \in \mathbf{N} \mid 1 \leq m \leq n \text{ and } (m, n) = 1\} = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right),$$

where the product is taken over the distinct primes p dividing n. Throughout this paper p denotes a prime. It is well known that for $n \in \mathbf{N}$,

$$2 \nmid \phi(n) \quad \Longleftrightarrow \quad n = 1, 2.$$

We are interested in those $n \in \mathbf{N}$ for which $q \nmid \phi(n)$, where q is a fixed odd prime. We set

$$(2) \qquad E_q(x) = \operatorname{card}\{n \leq x \mid q \nmid \phi(n)\}.$$

In 1990 Erdős, Granville, Pomerance and Spiro gave an upper bound for $E_q(x)$, which is valid for all sufficiently large x, see [1, Equation (4.2) with $k=1$, p. 191].

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In this paper we give an asymptotic formula for $E_q(x)$ as $x\to\infty$, see the theorem in Section 4. Let $0<\varepsilon<1$. For q a fixed odd prime, we show that

$$E_q(x)=e(q)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, where $e(q)$ is given in Definition 4.1 and the constant implied by the O-symbol depends only on q and ε. In 2002 Luca and Pomerance [2, Lemma 2, p. 114] proved the related result: For some constant $c>0$, for almost all n, $\phi(n)$ is divisible by all prime powers $p^a\leq c\log\log n/\log\log\log n$.

2. Notation

We denote the sets of real numbers and complex numbers by $\mathbf{R}$ and $\mathbf{C}$, respectively. As usual Γ denotes the gamma function and γ is Euler's constant. If K is an algebraic number field we write $h(K)$ for the class number of K and $R(K)$ for the regulator of K, see for example [3, pp. 97, 106]. Throughout this paper q denotes a fixed odd prime. We set

$$K_q:=\mathbf{Q}(e^{2\pi i/q})\subseteq\mathbf{C}, \tag{3}$$

so that K_q is a cyclotomic field with $[K_q:\mathbf{Q}]=\phi(q)=q-1$. For brevity we set

$$h(q):=h(K_q)\quad\text{and}\quad R(q):=R(K_q). \tag{4}$$

We also let

$$\omega:=e^{2\pi i/(q-1)}\in\mathbf{C}, \tag{5}$$

so that $\omega^{q-1}=1$. The principal character $\chi_0 \pmod q$ is defined as follows: for $n\in\mathbf{Z}$ we have

$$\chi_0(n)=\begin{cases}1, & \text{if } n\not\equiv 0 \pmod q,\\ 0, & \text{if } n\equiv 0 \pmod q.\end{cases} \tag{6}$$

Let g be a primitive root $\pmod q$. For $n\in\mathbf{Z}$ with $n\not\equiv 0 \pmod q$ the index $\mathrm{ind}_g(n)$ of n with respect to g is defined modulo $q-1$ by

$$n\equiv g^{\mathrm{ind}_g(n)} \pmod q.$$

We define a character $\chi_g \pmod q$ as follows: for $n\in\mathbf{Z}$ we set

$$\chi_g(n)=\begin{cases}\omega^{\mathrm{ind}_g n}, & \text{if } n\not\equiv 0 \pmod q,\\ 0, & \text{if } n\equiv 0 \pmod q.\end{cases} \tag{7}$$

There are exactly $\phi(q)=q-1$ characters $\pmod q$. They are

$$\chi_0,\ \chi_g,\ \chi_g^2,\dots,\ \chi_g^{q-2}, \tag{8}$$

where $\chi_g^{q-1}=\chi_0$.

3. The constant $C(q)$

It is convenient to define the following constant involving χ_g.

Definition 3.1. Let q be an odd prime. Let g be a primitive root $\pmod q$. Let $r\in\{1,2,\dots,q-2\}$. We define

$$C(q,r,\chi_g):=\prod_{\chi_g(p)=\omega^r}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right), \tag{9}$$

where the product is taken over all primes p such that $\chi_g(p)=\omega^r$.

Note that the prime q is not included in the product as $\chi_g(q)=0$ by (7). As $1\le(r,q-1)\le\frac{1}{2}(q-1)$ for $r\in\{1,2,\dots,q-2\}$ we have

$$\frac{q-1}{(r,q-1)}\ge 2 \tag{10}$$

so that the infinite product in (9) converges. Let h be another primitive root $\pmod q$. Then there exists an integer s such that

$$h\equiv g^s \pmod q, \quad (s,q-1)=1.$$

Let t be an integer such that $st\equiv 1 \pmod{q-1}$. Then, for $n\in\mathbf{N}$ with $n\not\equiv 0 \pmod q$, we have

$$\operatorname{ind}_h(n)\equiv t\operatorname{ind}_g(n) \pmod{q-1}$$

so that

$$\chi_h(n)=\omega^{\operatorname{ind}_h(n)}=\omega^{t\operatorname{ind}_g(n)}=(\chi_g(n))^t=\chi_g^t(n),$$

that is $\chi_h=\chi_g^t$. Hence

$$\begin{aligned}\prod_{r=1}^{q-2}C(q,r,\chi_h)^{(r,q-1)}&=\prod_{r=1}^{q-2}\prod_{\chi_h(p)=\omega^r}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}\\&=\prod_{r=1}^{q-2}\prod_{\chi_g^t(p)=\omega^r}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}\\&=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^{rs}}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}\end{aligned}$$

$$=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^{rs}}\left(1-\frac{1}{p^{(q-1)/(rs,q-1)}}\right)^{(rs,q-1)}$$

$$=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^{r}}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}$$

$$=\prod_{r=1}^{q-2}C(q,r,\chi_g)^{(r,q-1)}$$

so that the product

$$\prod_{r=1}^{q-2}C(q,r,\chi_g)^{(r,q-1)} \tag{11}$$

does not depend on the choice of primitive root g. Thus we can make the following definition.

Definition 3.2. Let q be an odd prime. We define the constant $C(q)$ by

$$C(q):=\prod_{r=1}^{q-2}C(q,r,\chi_g)^{(r,q-1)}. \tag{12}$$

We take this opportunity to determine $C(3)$. It is convenient to define the constant $k_{a,b}(m)$ by

$$k_{a,b}(m):=\prod_{p\equiv b\ (\mathrm{mod}\ a)}\left(1-\frac{1}{p^m}\right), \tag{13}$$

where $a\in\mathbf{N}$ and $b\in\mathbf{N}\cup\{0\}$ are such that $0\le b<a$ and $(a,b)=1$ and $m\in\mathbf{N}$ is such that $m\ge 2$.

Lemma 3.1. $C(3)=k_{3,2}(2)$.

Proof. Let $q=3$. Then $\omega=-1$, $r=1$, $g=2$ and $\chi_2(n)=(-3/n)$. Hence

$$C(3)=C(3,1,\chi_2)=\prod_{\chi_2(p)=-1}\left(1-\frac{1}{p^2}\right)=\prod_{p\equiv 2\ (\mathrm{mod}\ 3)}\left(1-\frac{1}{p^2}\right)=k_{3,2}(2),$$

as asserted. □

4. Statement of main result

We begin with a definition.

Definition 4.1. Let q be an odd prime. We define

$$e(q) := \frac{(q+1)(q-1)^{(q-2)/(q-1)}\Gamma\left(\frac{1}{q-1}\right)\sin\left(\frac{\pi}{q-1}\right)}{2^{(q-3)/2(q-1)}q^{3(q-2)/2(q-1)}\pi^{3/2}(h(q)R(q)C(q))^{1/(q-1)}}. \tag{14}$$

Before stating our main result, we give the value of $e(3)$.

Lemma 4.1.

$$e(3) = \frac{2^{7/2}}{3^{9/4}}k_{3,1}(2)^{1/2}.$$

Proof. We have $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $C(3)=k_{3,2}(2)$ and $h(3)=R(3)=1$, so that Definition 4.1 with $q=3$ gives

$$e(3) = \frac{2^{5/2}}{3^{3/4}\pi k_{3,2}(2)^{1/2}}.$$

As

$$\left(1-\frac{1}{3^2}\right)k_{3,1}(2)k_{3,2}(2) = \prod_p\left(1-\frac{1}{p^2}\right) = \frac{6}{\pi^2}$$

we have

$$k_{3,2}(2) = \frac{27}{4\pi^2}\frac{1}{k_{3,1}(2)} \quad \text{and} \quad e(3) = \frac{2^{7/2}}{3^{9/4}}k_{3,1}(2)^{1/2},$$

as asserted. □

Our main result is the following asymptotic formula for $E_q(x)$.

Theorem. *Let* $0<\varepsilon<1$. *For* q *an odd prime, we have*

$$E_q(x) = e(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, *where the constant implied by the* O*-symbol depends only on* q *and* ε, *and* $e(q)$ *is given in Definition* 4.1.

This theorem is proved in Section 7 after some preliminary results are given in Sections 5 and 6.

5. Preliminary results

The following results will be used in Sections 6 and 7.

Proposition 5.1. *Let $n \in \mathbf{N}$ and let q be an odd prime. Then*

$$q \nmid \phi(n) \iff n = \prod_{p \not\equiv 1 \pmod q} p^{a(p)} \text{ or } n = q \prod_{p \not\equiv 1 \pmod q} p^{a(p)},$$

where the product is taken over all primes $p \neq q$ with $p \not\equiv 1 \pmod q$ and the $a(p)$ are non-negative integers.

Proof. If

$$n = q^a \prod_{j=1}^{t} p_j^{a_j},$$

where a and t are non-negative integers, the p_j are distinct primes $\neq q$, and the a_j are non-negative integers, then by (1)

$$\phi(n) = \begin{cases} \prod_{j=1}^{t} p_j^{a_j - 1}(p_j - 1), & \text{if } a = 0, \\ q^{a-1}(q-1) \prod_{j=1}^{t} p_j^{a_j - 1}(p_j - 1), & \text{if } a \geq 1. \end{cases}$$

Hence $q \nmid \phi(n) \Leftrightarrow a \in \{0, 1\}$ and $q \nmid p_j - 1$ $(j = 1, \ldots, t)$, which proves Proposition 5.1. □

Next we define the set A by

(15) $$A = \{m \in \mathbf{N} \mid p \text{ (prime)} \mid m \Rightarrow p \neq q \text{ and } p \not\equiv 1 \pmod q\}.$$

The function $A(x)$ is defined for $x \in \mathbf{R}$ by

(16) $$A(x) = \sum_{\substack{m \leq x \\ m \in A}} 1.$$

Proposition 5.2. *For $x \in \mathbf{R}$ and q an odd prime we have*

$$E_q(x) = A(x) + A\left(\frac{x}{q}\right).$$

Proof. This follows immediately from Proposition 5.1. □

Proposition 5.3. (Wirsing's theorem) *Let $f\colon \mathbf{N}\to\mathbf{R}$ be multiplicative with $f(n)\geq 0$ for all $n\in\mathbf{N}$. Suppose that there exist constants c_1 and c_2 with $c_1>0$ and $0<c_2<2$ such that*

$$0\leq f(p^k)\leq c_1 c_2^k,$$

for all primes p and all $k\in\mathbf{N}$, and also that there is a constant τ with $\tau>0$ such that

$$\sum_{p\leq x} f(p)=\tau\frac{x}{\log x}+o\left(\frac{x}{\log x}\right),$$

as $x\to\infty$, then

$$\sum_{n\leq x} f(n)=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)\frac{x}{\log x}\prod_{p\leq x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right),$$

as $x\to\infty$.

Proof. See [7, Satz 1, p. 76]. □

Proposition 5.4. (Odoni's theorem) *Let $f\colon \mathbf{N}\to\mathbf{R}$ be multiplicative with $f(n)\geq 0$ for all $n\in\mathbf{N}$. Suppose that there exist constants $a_1>1$ and $a_2>1$ such that*

$$0\leq f(p^k)\leq a_1 k^{a_2},$$

for all primes p and all $k\in\mathbf{N}$, and also that there are constants τ and β with $\tau>0$ and $0<\beta<1$ such that

$$\sum_{p\leq x} f(p)=\tau\frac{x}{\log x}+O\left(\frac{x}{(\log x)^{1+\beta}}\right),$$

as $x\to\infty$, then there is a constant $B>0$ such that

$$\sum_{n\leq x} f(n)n^{-1}=B(\log x)^{\tau}+O((\log x)^{\tau-\beta}),$$

as $x\to\infty$. Further, for each fixed $\lambda>0$, we have

$$\sum_{n\leq x} f(n)n^{\lambda-1}=\lambda^{-1}Bx^{\lambda}\tau(\log x)^{\tau-1}+O(x^{\lambda}(\log x)^{\tau-1-\beta}), \tag{17}$$

as $x\to\infty$.

Proof. See [4, Theorem II, p. 205; Theorem III, p. 206; Note added in proof, p. 216]. □

From Propositions 5.3 and 5.4 we obtain the following corollary.

Proposition 5.5. *Let $f\colon \mathbf{N}\to\mathbf{R}$ be multiplicative with $0\le f(n)\le 1$ for all $n\in\mathbf{N}$. Suppose that there are constants τ and β with $\tau>0$ and $0<\beta<1$ such that*

$$\sum_{p\le x} f(p)=\tau\frac{x}{\log x}+O\left(\frac{x}{(\log x)^{1+\beta}}\right).$$

Then

$$\lim_{x\to\infty}\frac{1}{(\log x)^{\tau}}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right)$$

exists, and

$$\sum_{n\le x} f(n)=Ex(\log x)^{\tau-1}+O(x(\log x)^{\tau-1-\beta}),$$

with

$$E=\frac{e^{-\gamma\tau}}{\Gamma(\tau)}\lim_{x\to\infty}\frac{1}{(\log x)^{\tau}}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right).$$

Proof. The conditions of Odoni's theorem are met (with $a_1=a_2=2$) so by (17) with $\lambda=1$ there is a constant $B>0$ such that

$$\sum_{n\le x} f(n)=Bx\tau(\log x)^{\tau-1}+O(x(\log x)^{\tau-1-\beta}).$$

The conditions of Wirsing's theorem are also met (with $c_1=c_2=1$) so that

$$\sum_{n\le x} f(n)=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)\frac{x}{\log x}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right).$$

Equating the two expressions for $\sum_{n\le x} f(n)$, and dividing by $x(\log x)^{\tau-1}$, we obtain

$$B\tau+O((\log x)^{-\beta})=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)(\log x)^{-\tau}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right).$$

Letting $x\to\infty$ we have

$$\lim_{x\to\infty}(\log x)^{-\tau}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right)=B\tau\Gamma(\tau)e^{\gamma\tau}.$$

Thus

$$\sum_{n\le x} f(n) = Ex(\log x)^{\tau-1} + O(x(\log x)^{\tau-1-\beta}),$$

with

$$E = B\tau = \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \lim_{x\to\infty} (\log x)^{-\tau} \prod_{p\le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right),$$

as asserted. □

Proposition 5.6. *Let $k\in\mathbf{N}$ and $l\in\mathbf{N}$ be such that $1\le l\le k$ and $(k,l)=1$. Then*

$$\sum_{\substack{p\le x \\ p\equiv l \pmod k}} 1 = \frac{1}{\phi(k)} \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right), \quad \textit{as } x\to\infty.$$

Proof. This is the prime number theorem for the arithmetic progression $\{kr+l \mid r=0,1,2,\dots\}$, see for example [5, p. 139]. □

Let $k\in\mathbf{N}$. Let χ be a character $(\operatorname{mod} k)$. Let χ_0 be the principal character $(\operatorname{mod} k)$. The Dirichlet L-series corresponding to χ is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{18}$$

where $s=\sigma+it\in\mathbf{C}$. For $\chi\ne\chi_0$, the series in (18) converges for $\sigma>0$ and

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1} \ne 0. \tag{19}$$

For each character χ (mod k) we define a completely multiplicative function $k_\chi(n)$ $(n\in\mathbf{N})$ by setting, for primes p,

$$k_\chi(p) = p\left[1 - \left(1 - \frac{\chi(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-\chi(p)}\right]. \tag{20}$$

The Dirichlet series corresponding to k_χ is given by

$$K(s,\chi) = \sum_{n=1}^{\infty} \frac{k_\chi(n)}{n^s}, \tag{21}$$

where $s=\sigma+it\in\mathbf{C}$. It is shown in [6] that the series in (21) converges absolutely for $\sigma>0$ and that

$$K(1,\chi) = \sum_{n=1}^{\infty} \frac{k_\chi(n)}{n} = \prod_p \left(1 - \frac{k_\chi(p)}{p}\right)^{-1} = \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{\chi(p)} \ne 0.$$

Proposition 5.7. *Let $k\in\mathbf{N}$ and $l\in\mathbf{N}$ be such that $1\leq l\leq k$ and $(l,k)=1$. Then*

$$\prod_{\substack{p\leq x\\ p\equiv l \pmod k}}\left(1-\frac{1}{p}\right)=A(l,k)(\log x)^{-1/\phi(k)}+O((\log x)^{-1/\phi(k)-1}),$$

as $x\to\infty$, where

$$A(l,k)=\left(e^{-\gamma}\frac{k}{\phi(k)}\prod_{\chi\neq\chi_0}\left(\frac{K(1,\chi)}{L(1,\chi)}\right)^{\bar\chi(l)}\right)^{1/\phi(k)}.$$

Proof. This proposition is Mertens' theorem for the arithmetic progression $\{kr+l|r=0,1,2,...\}$, which was first proved by Williams [6] in 1974. □

Proposition 5.8. *Let $k,m,r\in\mathbf{N}$. Let ω_k be a primitive k-th root of unity. Then*

$$\prod_{j=0}^{k-1}\left(1-\frac{\omega_k^{jr}}{m}\right)=\left(1-\frac{1}{m^{k/(k,r)}}\right)^{(k,r)}.$$

Proof. Let $k,r\in\mathbf{N}$. Set

$$h=\frac{k}{(k,r)}\quad\text{and}\quad s=\frac{r}{(k,r)}.$$

As $(h,s)=1$ the h-th roots of unity are ω_h^{js}, $j=0,1,...,h-1$. Thus ω_h^{js}, $j=0,1,...,k-1$ comprise the h-th roots of unity each repeated k/h times. Hence

$$(x^h-1)^{k/h}=\prod_{j=0}^{k-1}(x-\omega_h^{js}).$$

Taking $x=m\in\mathbf{N}$, and dividing both sides by m^k, we obtain

$$\left(1-\frac{1}{m^h}\right)^{k/h}=\prod_{j=0}^{k-1}\left(1-\frac{\omega_h^{js}}{m}\right)=\prod_{j=0}^{k-1}\left(1-\frac{\omega_k^{jr}}{m}\right),$$

which is the asserted result. □

6. Estimation of $\prod_{\substack{p\leq x\\ p\equiv 1 \pmod q}}(1-1/p)$

We begin with the following result.

Proposition 6.1.

$$\prod_{j=1}^{q-2}K(1,\chi_g^j)=\frac{1}{C(q)}.$$

Proof. By Definition 3.1 we have

$$\prod_{r=1}^{q-2} C(q,r,\chi_g)^{-(r,q-1)} = \lim_{x\to\infty} \prod_{r=1}^{q-2} \prod_{\substack{p\le x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)}.$$

Next, as

$$\sum_{j=0}^{q-2} \omega^{jr} = \begin{cases} q-1, & \text{if } r=0, \\ 0, & \text{if } r=1,2,\dots,q-2, \end{cases}$$

we have

$$\prod_{r=1}^{q-2} \prod_{\substack{p\le x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)}$$

$$= \prod_{r=0}^{q-2} \prod_{\substack{p\le x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)} \left(1-\frac{1}{p}\right)^{\sum_{j=0}^{q-2}\omega^{jr}}.$$

By Proposition 5.8 with $m=p$, $k=q-1$ and $\omega=\omega_{q-1}$ we have

$$\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)} = \prod_{j=0}^{q-2}\left(1-\frac{\omega^{jr}}{p}\right)$$

so that

$$\prod_{r=0}^{q-2} \prod_{\substack{p\le x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)} \left(1-\frac{1}{p}\right)^{\sum_{j=0}^{q-2}\omega^{jr}}$$

$$= \prod_{r=0}^{q-2} \prod_{\substack{p\le x \\ \chi_g(p)=\omega^r}} \prod_{j=0}^{q-2} \left(1-\frac{\omega^{jr}}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\omega^{jr}}$$

$$= \prod_{j=1}^{q-2} \prod_{r=0}^{q-2} \prod_{\substack{p\le x \\ \chi_g(p)=\omega^r}} \left(1-\frac{\omega^{jr}}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\omega^{jr}}$$

$$= \prod_{j=1}^{q-2} \prod_{p\le x} \left(1-\frac{\chi_g^j(p)}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\chi_g^j(p)}.$$

Finally by Definition 3.2 we obtain

$$\frac{1}{C(q)} = \prod_{r=1}^{q-2} C(q,r,\chi_g)^{-(r,q-1)} = \prod_{j=1}^{q-2} \lim_{x\to\infty} \prod_{p\le x} \left(1-\frac{\chi_g^j(p)}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\chi_g^j(p)}$$
$$= \prod_{j=1}^{q-2} \prod_{p} \left(1-\frac{\chi_g^j(p)}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\chi_g^j(p)} = \prod_{j=1}^{q-2} K(1,\chi_g^j),$$

as asserted. □

Proposition 6.2.

$$\prod_{j=1}^{q-2} L(1,\chi_g^j) = 2^{(q-3)/2} q^{-q/2} \pi^{(q-1)/2} h(q) R(q).$$

Proof. The cyclotomic field K_q is a totally complex field which contains exactly $2q$ roots of unity, namely $\{\pm 1, \pm\omega_q, \pm\omega_q^2, ..., \pm\omega_q^{q-1}\}$. Hence, by the class number formula for abelian fields applied to the cyclotomic field K_q, we have

$$h(q)R(q) = 2q|d(K_q)|^{1/2} 2^{-(q-1)/2} \pi^{-(q-1)/2} \prod_{j=1}^{q-2} L(1,\chi_g^j),$$

where $d(K_q)$ is the discriminant of K_q, see for example [3, Theorem 8.4, p. 436]. Now the discriminant of K_q is given by

$$d(K_q) = (-1)^{q(q-1)/2} q^{q-2},$$

see for example [3, Theorem 2.9, p. 63]. Hence

$$\prod_{j=1}^{q-2} L(1,\chi_g^j) = 2^{(q-3)/2} q^{-q/2} \pi^{(q-1)/2} h(q) R(q),$$

as asserted. □

Proposition 6.3. *Let q be an odd prime. Then*

$$\prod_{\substack{p\le x \\ p\equiv 1 \pmod q}} \left(1-\frac{1}{p}\right) = \lambda(q)(\log x)^{-1/(q-1)} + O((\log x)^{-q/(q-1)}),$$

as $x\to\infty$, where

$$\lambda(q) = \left(\frac{e^{-\gamma} 2^{-(q-3)/2} q^{(q+2)/2} \pi^{-(q-1)/2}}{(q-1)h(q)R(q)C(q)}\right)^{1/(q-1)}.$$

Proof. By Propositions 6.1 and 6.2 we obtain

$$\prod_{j=1}^{q-2} \frac{K(1,\chi_g^j)}{L(1,\chi_g^j)} = \frac{2^{-(q-3)/2} q^{q/2} \pi^{-(q-1)/2}}{h(q)R(q)C(q)}.$$

By Proposition 5.7 with $k=q$ and $l=1$, we have

$$\prod_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, q)}} \left(1-\frac{1}{p}\right) = \lambda(q)(\log x)^{-1/(q-1)} + O((\log x)^{-q/(q-1)}),$$

where

$$\begin{aligned}\lambda(q) = A(1,q) &= \left(e^{-\gamma}\frac{q}{q-1}\prod_{j=1}^{q-2}\frac{K(1,\chi_g^j)}{L(1,\chi_g^j)}\right)^{1/(q-1)}\\ &= \left(e^{-\gamma}\frac{q}{q-1}\,\frac{2^{-(q-3)/2}q^{q/2}\pi^{-(q-1)/2}}{h(q)R(q)C(q)}\right)^{1/(q-1)}\\ &= \left(\frac{e^{-\gamma}2^{-(q-3)/2}q^{(q+2)/2}\pi^{-(q-1)/2}}{(q-1)h(q)R(q)C(q)}\right)^{1/(q-1)},\end{aligned}$$

as asserted. □

Proposition 6.4. *Let* $0<\varepsilon<1$. *Then*

$$A(x) = \alpha(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, *where*

$$\alpha(q) = \frac{(q-1)^{(q-2)/(q-1)}\Gamma\left(\frac{1}{q-1}\right)\sin\left(\frac{\pi}{q-1}\right)}{2^{(q-3)/2(q-1)}q^{(q-4)/2(q-1)}\pi^{3/2}(h(q)R(q)C(q))^{1/(q-1)}}.$$

(The constant implied by the O-symbol depends only on q and ε.*)*

Proof. By (16) we have

$$A(x) = \sum_{\substack{n\le x\\ n\in A}} 1 = \sum_{n\le x} f(n),$$

where

$$f(n) = \begin{cases} 1, & \text{if } n\in A,\\ 0, & \text{if } n\notin A.\end{cases}$$

Clearly $f(n)$ is a multiplicative function by (15). Moreover $0\leq f(n)\leq 1$ for all $n\in\mathbf{N}$. By Proposition 5.6 we have

$$\sum_{p\leq x} f(p)=\sum_{\substack{p\leq x\\ p\in A}} 1=\sum_{\substack{p\leq x\\ p\not\equiv 1 \pmod q}} 1+O(1)=\frac{q-2}{q-1}\frac{x}{\log x}+O\left(\frac{x}{(\log x)^2}\right),$$

as $x\to\infty$. Hence, by Proposition 5.5 with $\tau=(q-2)/(q-1)$ and $\beta=1-\varepsilon$, the limit

$$\lim_{x\to\infty}\frac{1}{(\log x)^{(q-2)/(q-1)}}\prod_{\substack{p\leq x\\ p\neq q\\ p\not\equiv 1 \pmod q}}\left(1-\frac{1}{p}\right)^{-1}$$

exists, say equal to $M(q)$, and

$$A(x)=\frac{e^{-\gamma(q-2)/(q-1)}}{\Gamma\left(\dfrac{q-2}{q-1}\right)}M(q)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$. Now for $x\geq q$

$$\prod_{\substack{p\leq x\\ p\neq q\\ p\not\equiv 1 \pmod q}}\left(1-\frac{1}{p}\right)=\frac{\displaystyle\prod_{p\leq x}\left(1-\frac{1}{p}\right)}{\left(1-\dfrac{1}{q}\right)\displaystyle\prod_{\substack{p\leq x\\ p\equiv 1 \pmod q}}\left(1-\frac{1}{p}\right)}.$$

By Mertens' theorem we have

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)=e^{-\gamma}(1+o(1))\frac{1}{\log x},$$

as $x\to\infty$, so appealing to Proposition 6.3, we obtain

$$\prod_{\substack{p\leq x\\ p\neq q\\ p\not\equiv 1 \pmod q}}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}(1+o(1))(\log x)^{-1}}{\left(1-\dfrac{1}{q}\right)\lambda(q)(1+o(1))(\log x)^{-1/(q-1)}}$$

$$=\frac{qe^{-\gamma}}{(q-1)\lambda(q)}(1+o(1))(\log x)^{-(q-2)/(q-1)},$$

so that

$$\frac{1}{(\log x)^{(q-2)/(q-1)}} \prod_{\substack{p\le x\\ p\ne q\\ p\not\equiv 1 \,(\mathrm{mod}\, q)}} \left(1-\frac{1}{p}\right)^{-1} = \frac{(q-1)e^{\gamma}\lambda(q)}{q}(1+o(1)).$$

Hence

$$M(q) = \frac{(q-1)e^{\gamma}\lambda(q)}{q}.$$

Finally

$$A(x) = \frac{e^{\gamma/(q-1)}}{\Gamma\left(\dfrac{q-2}{q-1}\right)} \frac{(q-1)}{q}\lambda(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, so that as

$$\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(\frac{q-2}{q-1}\right) = \frac{\pi}{\sin\dfrac{\pi}{q-1}}$$

we have

$$\alpha(q) = \frac{e^{\gamma/(q-1)}}{\Gamma\left(\dfrac{q-2}{q-1}\right)} \frac{(q-1)}{q}\lambda(q) = \frac{(q-1)^{(q-2)/(q-1)}\Gamma\left(\dfrac{1}{q-1}\right)\sin\left(\dfrac{\pi}{q-1}\right)}{2^{(q-3)/2(q-1)}q^{(q-4)/2(q-1)}\pi^{3/2}(h(q)R(q)C(q))^{1/(q-1)}}.$$

This completes the proof of Proposition 6.4. □

7. Proof of the theorem

By Propositions 5.2 and 6.4 we have

$$\begin{aligned} E_q(x) &= A(x) + A\left(\frac{x}{q}\right) \\ &= \alpha(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}) \\ &\quad + \alpha(q)\frac{x}{q}\left(\log\frac{x}{q}\right)^{-1/(q-1)} + O\left(\frac{x}{q}\left(\log\frac{x}{q}\right)^{-q/(q-1)+\varepsilon}\right) \\ &= \alpha(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}) \\ &\quad + \frac{\alpha(q)}{q}x((\log x)^{-1/(q-1)} + O((\log x)^{-q/(q-1)})) + O(x(\log x)^{-q/(q-1)+\varepsilon}) \end{aligned}$$

$$
\begin{aligned}
&= \alpha(q)\left(1+\frac{1}{q}\right)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon}) \\
&= e(q)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon}),
\end{aligned}
$$

as $x\to\infty$. □

By Lemma 4.1 and the theorem (with $q=3$), the number of $n\leq x$ for which $3\nmid\phi(n)$ is

$$
\frac{2^{7/2}}{3^{9/4}}\left(\prod_{p\equiv 1\,(\mathrm{mod}\,3)}\left(1-\frac{1}{p^2}\right)\right)^{1/2}x(\log x)^{-1/2}+O_\varepsilon(x(\log x)^{-3/2+\varepsilon}),
$$

as $x\to\infty$, for any $\varepsilon>0$.

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