An arithmetic formula of Liouville type and an extension of an identity of Ramanujan

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1 Introduction

Let $\theta \in \mathbb{R}$ and let $q \in \mathbb{C}$ be such that $|q| < 1$. Ramanujan [4] has proved the following identity

$$\left(\sum_{n=1}^{\infty} \sin n\theta \frac{q^n}{1-q^n}\right)^2 = \sum_{n=1}^{\infty} \frac{n-1}{2} \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} a(q,n) \cos n\theta \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \sum_{k=1}^{n} b(q,n,k) \cos k\theta \frac{q^n}{1-q^n},$$

where

$$a(q,n) = \frac{1}{1-q^n} - \frac{n-1}{2}, \quad b(q,n,k) = -1.$$

In this paper we prove the following analogous identity.

Theorem 1.1.

$$\left(\sum_{n=1}^{\infty} \sin n\theta \frac{q^n}{1-q^n}\right)^3 = \sum_{n=1}^{\infty} A(q,n) \sin n\theta \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} B(q,n,k) \sin k\theta \frac{q^n}{1-q^n},$$

where

$$A(q,n) = \frac{3}{2} \sum_{t=1}^{\infty} \frac{tq^t}{1-q^t} + \frac{3}{4} \frac{nq^n}{1-q^n} - \frac{3}{2} \frac{q^n}{(1-q^n)^2} - \frac{(n^2-3n+2)}{8}$$

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and

$$B(q,n,k) = \frac{3}{4}k + \frac{3}{2} \frac{1}{1-q^n} - \frac{3}{2}n.$$

We deduce this identity from the following new arithmetic identity, which is similar to one stated by Liouville [1, p. 331].

Theorem 1.2. *Let n be a positive integer, and let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$
$$= \sum_{d|n} \left(\frac{d^2-3d+2}{2} + 3\left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right)\right) F(d)$$
$$+ 3\sum_{d|n}\sum_{1\leq k<d}\left(2d - \frac{2n}{d} - k\right) F(k) - 6 \sum_{\substack{n_1,n_2\in\mathbb{N}\\ n=n_1+n_2}} \sigma(n_1)\sum_{d|n_2} F(d),$$

where the sum on the left-hand side is taken over all $(a,b,c,x,y,z) \in \mathbb{N}^6$ such that $ax+by+cz = n$ and $\sigma(n_1)$ denotes the sum of the divisors of n_1.

In Sections 2-6 we prove some arithmetic lemmas from which we deduce Theorem 1.2 in Section 7. Theorem 1.1 is deduced from Theorem 1.2 in Section 8. Application of Theorem 1.2 to sums of six squares is made in [3].

2 The equation $kx+ey = n$ and the quantities L_1, L_2, L_3, M_1, M_2, M_3

In this section we are interested in the number of solutions $(x, e, y) \in \mathbb{N}^3$ of the equation $kx + ey = n$, where k, n are fixed positive integers and the variables x, e, y are required to satisfy certain inequalities.

Definition 2.1.

$$L_1 = \sum_{\substack{kx+ey=n\\ e<k}} 1,\ L_2 = \sum_{\substack{kx+ey=n\\ e>k}} 1,\ L_3 = \sum_{\substack{kx+ey=n\\ e=k}} 1,$$

$$M_1 = \sum_{\substack{kx+ey=n\\ x<y}} 1,\ M_2 = \sum_{\substack{kx+ey=n\\ x>y}} 1,\ M_3 = \sum_{\substack{kx+ey=n\\ x=y}} 1.$$

Thus, for fixed positive integers k and n, the sum L_1 counts the number of triples $(x, e, y) \in \mathbb{N}^3$ satisfying the equation $kx + ey = n$ with $e < k$. The following result follows immediately from Definition 2.1.

Lemma 2.1.

$$L_1 + L_2 + L_3 = M_1 + M_2 + M_3.$$

Next we evaluate L_3. We set $\delta(n, k) = 1$ if k divides n and 0 otherwise.

Lemma 2.2.

$$L_3 = \delta(n, k)\left(\frac{n}{k} - 1\right).$$

Proof. We have

$$L_3 = \sum_{\substack{kx+ey=n \\ e=k}} 1 = \sum_{x+y=n/k} 1 = \delta(n, k)\left(\frac{n}{k} - 1\right)$$

as asserted. □

M_3 can be determined in a similar manner.

Lemma 2.3.

$$M_3 = \sum_{\substack{d|n \\ d>k}} 1.$$

Our next lemma shows that L_2 and M_2 are equal.

Lemma 2.4.

$$L_2 = M_2.$$

Proof. We have

$$M_2 = \sum_{\substack{kx+ey=n \\ x>y}} 1 = \sum_{k(x+y)+ey=n} 1 = \sum_{kx+(k+e)y=n} 1 = \sum_{\substack{kx+ey=n \\ e>k}} 1 = L_2,$$

as asserted. □

Lemma 2.5.

$$L_1 - M_1 = \sum_{\substack{d|n \\ d>k}} 1 - \delta(n, k)\left(\frac{n}{k} - 1\right).$$

Proof. By Lemmas 2.1 and 2.4 we have $L_1 - M_1 = M_3 - L_3$ and the asserted result follows from Lemmas 2.2 and 2.3. $\square$

3 The equation $kx+ey=n$ and the quantities R_1, R_2, R_3, S_1, S_2, S_3

In this section we introduce the quantities R_1, R_2, R_3, S_1, S_2, S_3 formed from L_1, L_2, L_3, M_1, M_2, M_3 respectively by taking the sums over y rather than 1.

Definition 3.1.

$$R_1 = \sum_{\substack{kx+ey=n \\ e<k}} y, \; R_2 = \sum_{\substack{kx+ey=n \\ e>k}} y, \; R_3 = \sum_{\substack{kx+ey=n \\ e=k}} y,$$

$$S_1 = \sum_{\substack{kx+ey=n \\ x<y}} y, \; S_2 = \sum_{\substack{kx+ey=n \\ x>y}} y, \; S_3 = \sum_{\substack{kx+ey=n \\ x=y}} y.$$

The following result follows immediately from Definition 3.1.

Lemma 3.1.

$$R_1 + R_2 + R_3 = S_1 + S_2 + S_3.$$

Similarly to Lemmas 2.2, 2.3 and 2.4 respectively, we obtain

Lemma 3.2.

$$R_3 = \delta(n,k)\frac{(n-k)n}{2k^2}.$$

Lemma 3.3.

$$S_3 = \sum_{\substack{d|n \\ d>k}} \frac{n}{d}.$$

Lemma 3.4.

$$R_2 = S_2.$$

4 The equation $kx+ey=n$ and the quantities P_1, P_2, P_3, Q_1, Q_2, Q_3

In this section we introduce the quantities P_1, P_2, P_3, Q_1, Q_2, Q_3 formed from L_1, L_2, L_3, M_1, M_2, M_3 respectively by taking the sums over x rather than 1.

Definition 4.1.

$$P_1 = \sum_{\substack{kx+ey=n \\ e<k}} x, \ P_2 = \sum_{\substack{kx+ey=n \\ e>k}} x, \ P_3 = \sum_{\substack{kx+ey=n \\ e=k}} x,$$

$$Q_1 = \sum_{\substack{kx+ey=n \\ x<y}} x, \ Q_2 = \sum_{\substack{kx+ey=n \\ x>y}} x, \ Q_3 = \sum_{\substack{kx+ey=n \\ x=y}} x.$$

The following result follows immediately from Definition 4.1.

Lemma 4.1.

$$P_1 + P_2 + P_3 = Q_1 + Q_2 + Q_3.$$

Similarly to the results in Sections 2 and 3 we obtain the following results.

Lemma 4.2.

$$P_3 = \delta(n,k)\frac{(n-k)n}{2k^2}.$$

Lemma 4.3.

$$Q_3 = \sum_{\substack{d|n \\ d>k}} \frac{n}{d}.$$

Lemma 4.4.

$$Q_2 = P_2 + R_2.$$

Lemma 4.5.

$$P_1 - Q_1 - S_2 = \sum_{\substack{d|n \\ d>k}} \frac{n}{d} - \delta(n,k)\left(\frac{(n-k)n}{2k^2}\right).$$

5 Solutions of $kx + by + cz = n$ (k, n fixed integers)

In this section we are concerned with solutions $(x, b, y, c, z) \in \mathbb{N}^5$ of the equation $kx + by + cz = n$, where k and n are fixed positive integers. We begin by defining the following quantities.

Definition 5.1.

$$A_1 = \sum_{\substack{kx+by+cz=n \\ b+c<k}} 1, \qquad A_2 = \sum_{\substack{kx+by+cz=n \\ b+c>k}} 1, \qquad A_3 = \sum_{\substack{kx+by+cz=n \\ b+c=k}} 1.$$

Our next lemma evaluates A_3 in terms of L_1 and P_1.

Lemma 5.1.

$$A_3 = 2(P_1 - L_1) + (k-1)\left(\frac{n}{k} - 1\right)\delta(n, k).$$

Proof. We have

$$\begin{aligned}
A_3 &= \sum_{\substack{kx+by+cz=n \\ b+c=k \\ y<z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c=k \\ y>z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c=k \\ y=z}} 1 \\
&= \sum_{\substack{kx+(k-c)y+cz=n \\ c<k,\ y<z}} 1 + \sum_{\substack{kx+by+(k-b)z=n \\ b<k,\ y>z}} 1 + \sum_{\substack{kx+(b+c)y=n \\ b+c=k}} 1 \\
&= \sum_{\substack{k(x+y)+c(z-y)=n \\ c<k,\ y<z}} 1 + \sum_{\substack{k(x+z)+b(y-z)=n \\ b<k,\ y>z}} 1 + \sum_{k(x+y)=n} (k-1) \\
&= \sum_{\substack{k(x+y)+cz'=n \\ c<k}} 1 + \sum_{\substack{k(x+z)+bz'=n \\ b<k}} 1 + (k-1)\left(\frac{n}{k} - 1\right)\delta(n, k) \\
&= \sum_{\substack{kx'+cz'=n \\ c<k}} (x'-1) + \sum_{\substack{kx'+bz'=n \\ b<k}} (x'-1) + (k-1)\left(\frac{n}{k} - 1\right)\delta(n, k) \\
&= 2(P_1 - L_1) + (k-1)\left(\frac{n}{k} - 1\right)\delta(n, k),
\end{aligned}$$

by Definitions 2.1 and 4.1. $\square$

Next we define the following quantities.

Definition 5.2.

$$B_1 = \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x<z}} 1, \qquad B_2 = \sum_{\substack{kx+by+cz=n \\ b-c>k,\ x<z}} 1, \qquad B_3 = \sum_{\substack{kx+by+cz=n \\ b-c=k,\ x<z}} 1,$$

$$B_4 = \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x>z}} 1, \qquad B_5 = \sum_{\substack{kx+by+cz=n \\ b-c>k,\ x>z}} 1, \qquad B_6 = \sum_{\substack{kx+by+cz=n \\ b-c=k,\ x>z}} 1,$$

$$B_7 = \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x=z}} 1, \qquad B_8 = \sum_{\substack{kx+by+cz=n \\ b-c>k,\ x=z}} 1, \qquad B_9 = \sum_{\substack{kx+by+cz=n \\ b-c=k,\ x=z}} 1.$$

The next three lemmas relate B_3, B_4 and B_6 to quantities defined in Sections 2-4. We just give the proof of Lemma 5.4.

Lemma 5.2.

$$B_3 = Q_1 - M_1.$$

Lemma 5.3.

$$B_6 = S_2 - M_2.$$

Lemma 5.4.

$$A_2 = 2B_4 + \sum_{\substack{kx+ey=n \\ e>k}} e - L_2.$$

Proof. We have

$$\begin{aligned} A_2 &= \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y>z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y<z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y=z}} 1 \\ &= 2 \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y>z}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y=z}} 1. \end{aligned}$$

The last step is obtained by observing the symmetry between b and c, and between y and z. We now consider these two sums. Firstly,

$$\sum_{\substack{kx+by+cz=n \\ b+c>k,\ y=z}} 1 = \sum_{\substack{kx+(b+c)y=n \\ b+c>k}} 1 = \sum_{\substack{kx+ey=n \\ e>k \\ e=b+c}} 1 = \sum_{\substack{kx+ey=n \\ e>k}} (e-1) = \sum_{\substack{kx+ey=n \\ e>k}} e - L_2,$$

and secondly,

$$\begin{aligned} B_4 &\overset{x=z+x'}{=} \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x>z}} 1 = \sum_{\substack{k(z+x')+by+cz=n \\ b-c<k}} 1 = \sum_{\substack{kx+by+(k+c)z=n \\ b<k+c}} 1 \\ &\overset{e=k+c}{=} \sum_{\substack{kx+by+ez=n \\ b<e,\ e>k}} 1 \overset{e=b+c'}{=} \sum_{\substack{kx+by+(b+c')z=n \\ b+c'>k}} 1 = \sum_{\substack{kx+b(y+z)+cz=n \\ b+c>k}} 1 \\ &\overset{y'=y+z}{=} \sum_{\substack{kx+by'+cz=n \\ b+c>k,\ y'>z}} 1 = \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y>z}} 1. \end{aligned}$$

This completes the proof of the lemma. $\square$

We next define the following quantities. Our ultimate goal in this section is to determine $A_1 - 2B_1 + 2C_1$, see Proposition 5.1.

Definition 5.3.

$$C_1 = \sum_{\substack{kx+by+cz=n \\ x<y<z}} 1, \qquad C_2 = \sum_{\substack{kx+by+cz=n \\ x<z<y}} 1, \qquad C_3 = \sum_{\substack{kx+by+cz=n \\ y<x<z}} 1,$$

$$C_4 = \sum_{\substack{kx+by+cz=n \\ y<z<x}} 1, \qquad C_5 = \sum_{\substack{kx+by+cz=n \\ z<x<y}} 1, \qquad C_6 = \sum_{\substack{kx+by+cz=n \\ z<y<x}} 1,$$

$$C_7 = \sum_{\substack{kx+by+cz=n \\ x<y=z}} 1, \qquad C_8 = \sum_{\substack{kx+by+cz=n \\ y<x=z}} 1, \qquad C_9 = \sum_{\substack{kx+by+cz=n \\ z<x=y}} 1,$$

$$C_{10} = \sum_{\substack{kx+by+cz=n \\ x=y<z}} 1, \qquad C_{11} = \sum_{\substack{kx+by+cz=n \\ x=z<y}} 1, \qquad C_{12} = \sum_{\substack{kx+by+cz=n \\ y=z<x}} 1,$$

$$C_{13} = \sum_{\substack{kx+by+cz=n \\ x=y=z}} 1.$$

The following lemma follows immediately from Definitions 5.1, 5.2 and 5.3.

Lemma 5.5.

$$\sum_{i=1}^{3} A_i = \sum_{i=1}^{9} B_i = \sum_{i=1}^{13} C_i.$$

The equalities in the next lemma follow by symmetry.

Lemma 5.6.

$$C_1 = C_2, \qquad C_3 = C_5, \qquad C_4 = C_6, \qquad C_8 = C_9, \qquad C_{10} = C_{11}.$$

Lemma 5.7.

$$B_2 = C_3.$$

Proof. We have

$$\begin{aligned} C_3 &\overset{\substack{x=y+x' \\ z=y+z'}}{=} \sum_{\substack{k(y+x')+by+c(y+z')=n \\ x'<z'}} 1 \overset{b'=k+b+c}{=} \sum_{\substack{kx+(b+c+k)y+cz=n \\ x<z}} 1 \\ &= \sum_{\substack{kx+b'y+cz=n \\ x<z,\ b'>k+c}} = \sum_{\substack{kx+by+cz=n \\ x<z,\ b-c>k}} 1 = B_2, \end{aligned}$$

as claimed. □

Similarly to Lemma 5.7, we have

Lemma 5.8.

$$B_5 = C_4.$$

Lemma 5.9.

$$C_7 = \sum_{\substack{kx+ey=n \\ x<y}} e - M_1.$$

Proof. We have

$$C_7 = \sum_{\substack{kx+by+cz=n \\ x<y=z}} 1 = \sum_{\substack{kx+(b+c)y=n \\ x<y}} 1 = \sum_{e=b+c} \sum_{\substack{kx+ey=n \\ x<y}} 1$$

$$= \sum_{e} (e-1) \sum_{\substack{kx+ey=n \\ x<y}} 1 = \sum_{\substack{kx+ey=n \\ x<y}} e - \sum_{\substack{kx+ey=n \\ x<y}} 1 = \sum_{\substack{kx+ey=n \\ x<y}} e - M_1,$$

by Definition 2.1. □

Similarly to Lemma 5.9, we obtain

Lemma 5.10.

$$C_{12} = \sum_{\substack{kx+ey=n \\ x>y}} e - M_2.$$

Lemma 5.11.

$$C_{13} = \sum_{\substack{d|n \\ d>k}} (d-k-1).$$

Proof. We have

$$C_{13} = \sum_{\substack{kx+by+cz=n \\ x=y=z}} 1 = \sum_{(k+b+c)x=n} 1 = \sum_{\substack{(k+e)x=n \\ e=b+c}} 1 = \sum_{(k+e)x=n} (e-1)$$

$$= \sum_{\substack{d|n \\ d=k+e}} (e-1) = \sum_{\substack{d|n \\ d>k}} (d-k-1),$$

as asserted. □

Lemma 5.12.

$$B_7 + B_8 + B_9 = C_8 + C_{10} + C_{13}.$$

Proof. This follows easily from Definitions 5.2 and 5.3. □

Proposition 5.1.

$$A_1 - 2B_1 + 2C_1 = - \sum_{\substack{kx+ey=n \\ e>k}} e - \sum_{\substack{kx+ey=n \\ x\neq y}} e - (k-1)\left(\frac{n}{k} - 1\right) \delta(n,k)$$

$$+\sum_{\substack{d|n\\ d>k}}\left(d-2\frac{n}{d}-k-1\right)+\delta(n,k)\left(\frac{n}{k}-1\right)\left(\frac{n}{k}\right)-M_1+2L_1.$$

Proof. By Lemmas 5.2, 5.3, 5.7, 5.8 and 5.12, we obtain

$$\sum_{i=1}^{9}B_i=B_1+C_3+Q_1-M_1+B_4+C_4+S_2-M_2+C_8+C_{10}+C_{13}.$$

By Lemma 5.6 we have

$$\sum_{i=1}^{13}C_i=2C_1+2C_3+2C_4+2C_8+2C_{10}+C_7+C_{12}+C_{13}.$$

From Lemma 5.5 we deduce

$$\sum_{i=1}^{3}A_i-2\sum_{i=1}^{9}B_i+\sum_{i=1}^{13}C_i=0.$$

Substituting the expressions obtained above for $\sum_{i=1}^{9}B_i$ and $\sum_{i=1}^{13}C_i$ into this equation, we obtain after some cancellation

$$\begin{aligned}&A_1-2B_1+2C_1\\ &\quad=(-A_2+2B_4)-A_3-C_7-C_{12}+C_{13}+2Q_1-2M_1+2S_2-2M_2.\end{aligned}$$

Then, appealing to Lemmas 5.1, 5.4, 5.9, 5.10 and 5.11, we obtain

$$\begin{aligned}&A_1-2B_1+2C_1\\ &=-\sum_{\substack{kx+ey=n\\ e>k}}e+L_2-2(P_1-L_1)-(k-1)\left(\frac{n}{k}-1\right)\delta(n,k)-\sum_{\substack{kx+ey=n\\ x<y}}e+M_1\\ &\quad-\sum_{\substack{kx+ey=n\\ x>y}}e+M_2+\sum_{\substack{d|n\\ d>k}}(d-k-1)+2Q_1-2M_1+2S_2-2M_2\\ &=-\sum_{\substack{kx+ey=n\\ e>k}}e-\sum_{\substack{kx+ey=n\\ x\neq y}}e-(k-1)\left(\frac{n}{k}-1\right)\delta(n,k)+\sum_{\substack{d|n\\ d>k}}(d-k-1)\\ &\quad-2(P_1-Q_1-S_2)-M_1+L_2-M_2+2L_1\end{aligned}$$

$$= -\sum_{\substack{kx+ey=n\\ e>k}} e - \sum_{\substack{kx+ey=n\\ x\neq y}} e - (k-1)\left(\frac{n}{k}-1\right)\delta(n,k) + \sum_{\substack{d\mid n\\ d>k}}(d-k-1)$$
$$-2\left(\sum_{\substack{d\mid n\\ d>k}}\frac{n}{d} - \frac{1}{2}\delta(n,k)\left(\frac{n}{k}-1\right)\left(\frac{n}{k}\right)\right) - M_1 + 2L_1,$$

by Lemmas 2.4 and 4.5. Simplifying, we obtain the required assertion. □

6 Solutions of $ax+by+cz=n$ (n fixed integer)

Throughout this section k and n are fixed positive integers. We are interested in the number U (resp. V, W) of solutions $(a,x,b,y,c,z)\in\mathbb{N}^6$ of the equation $ax+by+cz=n$ with $a+b+c=k$ (resp. $a+b-c=k$, $a-b-c=k$).

Definition 6.1.

$$U = \sum_{\substack{ax+by+cz=n\\ a+b+c=k}} 1, \qquad V = \sum_{\substack{ax+by+cz=n\\ a+b-c=k}} 1, \qquad W = \sum_{\substack{ax+by+cz=n\\ a-b-c=k}} 1.$$

In the next three lemmas we relate U, V and W to quantities of the previous four sections.

Lemma 6.1.

$$U = \delta(n,k)\left(\frac{k^2-3k+2}{2}\right) - 3\sum_{\substack{kx+ey=n\\ e<k}} e + 3(k-1)L_1 + 3A_1.$$

Proof. We have

$$U = \sum_{\substack{ax+by+cz=n\\ a+b+c=k}} 1 = \sum_{\substack{ax+by+cz=n\\ a+b+c=k\\ x\neq y,y\neq z,z\neq x}} 1 + \sum_{\substack{ax+by+cz=n\\ a+b+c=k\\ \text{exactly 2 of } x,y,z \text{ equal}}} 1 + \sum_{\substack{ax+by+cz=n\\ a+b+c=k\\ x=y=z}} 1.$$

We begin by considering the first sum on the right-hand side of the above expression for U. The condition $x\neq y, y\neq z, z\neq x$, comprises six possibilities, namely, $x<y<z$, $x<z<y$, $y<x<z$, $y<z<x$, $z<x<y$

and $z < y < x$. Simple transformations of the summation variables show that these six cases yield the same contribution to the sum. Thus

$$\sum_{\substack{ax+by+cz=n\\a+b+c=k\\x\neq y,y\neq z,z\neq x}} 1 = 6\sum_{\substack{ax+by+cz=n\\a+b+c=k\\x<y<z}} 1 = 6\sum_{\substack{(k-b-c)x+by+cz=n\\b+c<k\\x<y<z}} 1 = 6\sum_{\substack{kx+b(y-x)+c(z-x)=n\\b+c<k\\x<y<z}} 1$$

$$\overset{\substack{y'=y-x\\z'=z-x}}{=} 6\sum_{\substack{kx+by'+cz'=n\\b+c<k\\y'<z'}} 1 = 3\sum_{\substack{kx+by+cz=n\\b+c<k\\y\neq z}} 1$$

$$= 3\sum_{\substack{kx+by+cz=n\\b+c<k}} 1 - 3\sum_{\substack{kx+by+cz=n\\b+c<k\\y=z}} 1$$

$$= 3A_1 - 3\sum_{\substack{kx+(b+c)y=n\\b+c<k}} 1 = 3A_1 - 3\sum_{\substack{kx+ey=n\\e<k}} (e-1)$$

$$= 3A_1 + 3L_1 - 3\sum_{\substack{kx+ey=n\\e<k}} e,$$

by Definition 2.1. Next we consider the second sum on the right-hand side of the expression for U. There are six different cases in which exactly two of x, y, and z are equal. Simple transformations of the summation variables show that the cases $x = y < z$, $x = z < y$, and $y = z < x$, lead to the same sum, and similarly for the cases $x < y = z$, $y < x = z$, and $z < y = x$. Therefore

$$\sum_{\substack{ax+by+cz=n\\a+b+c=k\\\text{exactly 2 of }x,y,z\text{ equal}}} 1 = 3\sum_{\substack{ax+by+cz=n\\a+b+c=k\\x=y<z}} 1 + 3\sum_{\substack{ax+by+cz=n\\a+b+c=k\\x<y=z}} 1$$

$$= 3\sum_{\substack{(a+b)x+cz=n\\a+b+c=k\\x<z}} 1 + 3\sum_{\substack{ax+(b+c)y=n\\a+b+c=k\\x<y}} 1$$

$$= 3\sum_{\substack{ex+cz=n\\e+c=k\\x<z\\e=a+b}} 1 + 3\sum_{\substack{ax+ey=n\\a+e=k\\x<y\\e=b+c}} 1$$

$$
\begin{aligned}
&= 3 \sum_{\substack{ex+cz=n \\ e+c=k \\ x<z}} (e-1) + 3 \sum_{\substack{ax+ey=n \\ a+e=k \\ x<y}} (e-1) \\
&= 3 \sum_{\substack{ex+(k-e)z=n \\ x<z \\ e<k}} (e-1) + 3 \sum_{\substack{(k-e)x+ey=n \\ x<y \\ e<k}} (e-1) \\
&= 3 \sum_{\substack{ex+(k-e)(x+z')=n \\ e<k}} (e-1) + 3 \sum_{\substack{(k-e)x+e(x+y')=n \\ e<k}} (e-1) \\
&= 3 \sum_{\substack{kx+(k-e)z=n \\ e<k}} (e-1) + 3 \sum_{\substack{kx+ey=n \\ e<k}} (e-1) \\
&= 3 \sum_{\substack{kx+e'z=n \\ e'<k}} (k-e'-1) + 3 \sum_{\substack{kx+ey=n \\ e<k}} (e-1) \\
&= 3(k-2)L_1,
\end{aligned}
$$

by Definition 2.1. Finally, we consider the third sum on the right-hand side of the expression for U. We have

$$
\sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x=y=z}} 1 = \sum_{\substack{kx=n \\ a+b+c=k}} 1 = \delta(n,k) \sum_{a+b+c=k} 1 = \delta(n,k) \left(\frac{k^2-3k+2}{2} \right).
$$

Putting these three sums together, we obtain

$$
\begin{aligned}
U &= 3A_1 + 3L_1 - 3 \sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-2)L_1 + \delta(n,k) \left(\frac{k^2-3k+2}{2} \right) \\
&= \delta(n,k) \left(\frac{k^2-3k+2}{2} \right) - 3 \sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-1)L_1 + 3A_1,
\end{aligned}
$$

as asserted. □

Lemma 6.2.

$$
V = 2B_1 + (k-1)M_1 + \sum_{\substack{kx+ey=n \\ x<y}} e.
$$

Proof. We have

$$\begin{aligned} V = \sum_{\substack{ax+by+cz=n \\ a+b-c=k}} 1 &= 2 \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x<y}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ x=y}} 1 \\ &= 2 \sum_{\substack{(k-b+c)x+by+cz=n \\ b-c<k \\ x<y}} 1 + \sum_{\substack{(a+b)x+cz=n \\ a+b-c=k}} 1 \\ &= 2 \sum_{\substack{kx+b(y-x)+c(z+x)=n \\ b-c<k \\ x<y}} 1 + \sum_{\substack{ex+cz=n \\ e-c=k \\ e=a+b}} 1 \\ &= 2 \sum_{\substack{kx+by'+c(z+x)=n \\ b-c<k}} 1 + \sum_{\substack{ex+cz=n \\ e=c+k}} (e-1) \\ &= 2 \sum_{\substack{kx+by+cz'=n \\ b-c<k,\ z'>x}} 1 + \sum_{(c+k)x+cz=n} (c+k-1) \\ &= 2 \sum_{\substack{kx+by+cz=n \\ b-c<k,\ z>x}} 1 + \sum_{\substack{kx+cy=n \\ y>x}} (k+c-1) \\ &= 2B_1 + (k-1)M_1 + \sum_{\substack{kx+ey=n \\ x<y}} e, \end{aligned}$$

by Definitions 2.1 and 5.2. □

Lemma 6.3.

$$W = 2C_1 + \sum_{\substack{kx+ey=n \\ x<y}} e - M_1.$$

Proof. We have

$$\begin{aligned} W = \sum_{\substack{ax+by+cz=n \\ a-b-c=k}} 1 &= \sum_{(k+b+c)x+by+cz=n} 1 = \sum_{kx+b(x+y)+c(x+z)=n} 1 \\ = \sum_{\substack{kx+by'+cz'=n \\ y'>x,\ z'>x}} 1 &= \sum_{\substack{kx+by+cz=n \\ y>x,\ z>x,\ y<z}} 1 + \sum_{\substack{kx+by+cz=n \\ y>x,\ z>x,\ y>z}} 1 + \sum_{\substack{kx+by+cz=n \\ y>x,\ z>x,\ y=z}} 1 \end{aligned}$$

$$= 2 \sum_{\substack{kx+by+cz=n \\ x<y<z}} 1 + \sum_{\substack{kx+(b+c)y=n \\ y>x}} 1 = 2C_1 + \sum_{\substack{kx+ey=n \\ y>x}} (e-1),$$

by Definition 5.3, as claimed. □

Lemma 6.4.

$$U - 3V + 3W = \delta(n,k)\left(\frac{k^2-3k+2}{2} + 3\left(\frac{n}{k}-1\right)\left(\frac{n}{k}-k\right)\right) + 3\sum_{\substack{d\mid n \\ d>k}}\left(2d - 2\frac{n}{d} - k\right) - 6\sum_{kx+ey=n} e.$$

Proof. Applying Lemmas 6.1, 6.2 and 6.3, we find

$$\begin{aligned} &U - 3V + 3W \\ &= \delta(n,k)\left(\frac{k^2-3k+2}{2}\right) - 3\sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-1)L_1 + 3A_1 \\ &\quad - 3\left(2B_1 + (k-1)M_1 + \sum_{\substack{kx+ey=n \\ x<y}} e\right) + 3\left(2C_1 + \sum_{\substack{kx+ey=n \\ x<y}} e - M_1\right) \\ &= \delta(n,k)\left(\frac{k^2-3k+2}{2}\right) - 3\sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-1)L_1 - 3kM_1 \\ &\quad + 3(A_1 - 2B_1 + 2C_1). \end{aligned}$$

Appealing to Proposition 5.1, we obtain

$$\begin{aligned} &U - 3V + 3W \\ &= \delta(n,k)\left(\frac{k^2-3k+2}{2}\right) - 3\sum_{\substack{kx+ey=n \\ e<k}} e + 3(k-1)L_1 - 3kM_1 \\ &\quad + 3\Bigg(-\sum_{\substack{kx+ey=n \\ e>k}} e - \sum_{\substack{kx+ey=n \\ x\neq y}} e - (k-1)(\frac{n}{k}-1)\delta(n,k) \end{aligned}$$

$$+ \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k - 1 \right) + \delta(n,k) \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} \right) - M_1 + 2L_1 \Bigg)$$

$$= \delta(n,k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k + 1 \right) \right) - 3 \sum_{\substack{kx+ey=n \\ e \neq k}} e - 3 \sum_{\substack{kx+ey=n \\ x \neq y}} e$$

$$+ 3(k+1)(L_1 - M_1) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k - 1 \right)$$

$$= \delta(n,k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k + 1 \right) \right) - 6 \sum_{kx+ey=n} e + 3 \sum_{(k+e)x=n} e$$

$$+ 3 \sum_{k(x+y)=n} k + 3(k+1) \left(\sum_{\substack{d|n \\ d>k}} 1 - \delta(n,k) \left(\frac{n}{k} - 1 \right) \right) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} - k - 1 \right)$$

(by Lemma 2.5)

$$= \delta(n,k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - 2k \right) \right) - 6 \sum_{kx+ey=n} e + 3 \sum_{\substack{d|n \\ d>k}} (d-k)$$

$$+ 3\delta(n,k)k \left(\frac{n}{k} - 1 \right) + 3 \sum_{\substack{d|n \\ d>k}} \left(d - 2\frac{n}{d} \right)$$

$$= \delta(n,k) \left(\frac{k^2 - 3k + 2}{2} + 3 \left(\frac{n}{k} - 1 \right) \left(\frac{n}{k} - k \right) \right) + 3 \sum_{\substack{d|n \\ d>k}} \left(2d - 2\frac{n}{d} - k \right) - 6 \sum_{kx+ey=n} e,$$

as required. □

7 Proof of Theorem 1.2.

We now use the results of Sections 2-6 to prove our formula of Liouville type (Theorem 1.1). First we have

$$\sum_{ax+by+cz=n} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$

$$= \sum_{ax+by+cz=n} F(a+b+c) + \sum_{ax+by+cz=n} F(a-b-c)$$
$$- \sum_{ax+by+cz=n} F(a+b-c) - \sum_{ax+by+cz=n} F(a-b+c).$$

Next we look at each of these sums individually. We have (recalling that F is odd)

$$\begin{aligned}
\sum_{ax+by+cz=n} F(a+b+c) &= \sum_{k\in\mathbb{N}} F(k) \sum_{\substack{ax+by+cz=n \\ a+b+c=k}} 1 = \sum_{k\in\mathbb{N}} F(k)U, \\
\sum_{ax+by+cz=n} F(a-b-c) &= \sum_{k\in\mathbb{N}} F(k)\left(W-V\right), \\
\sum_{ax+by+cz=n} F(a+b-c) &= \sum_{k\in\mathbb{N}} F(k)\left(V-W\right), \\
\sum_{ax+by+cz=n} F(a-b+c) &= \sum_{k\in\mathbb{N}} F(k)\left(V-W\right).
\end{aligned}$$

Therefore

$$\sum_{ax+by+cz=n} \left(F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)\right)$$
$$= \sum_{k\in\mathbb{N}} F(k)(U-3V+3W).$$

Appealing to Lemma 6.4, we obtain

$$\sum_{ax+by+cz=n} \left(F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c)\right)$$
$$= \sum_{k\in\mathbb{N}} F(k)\Bigg(\delta(n,k)\left(\frac{k^2-3k+2}{2} + 3\left(\frac{n}{k}-1\right)\left(\frac{n}{k}-k\right)\right)$$
$$+ 3\sum_{\substack{d\mid n \\ d>k}} \left(2d - 2\frac{n}{d} - k\right) F(k) - 6 \sum_{kx+ey=n} e\Bigg)$$
$$= \sum_{d\mid n} \left(\frac{d^2-3d+2}{2} + 3\left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right)\right) F(d)$$

$$+3\sum_{d|n}\sum_{1\le k<d}\left(2d-2\frac{n}{d}-k\right)F(k)-6\sum_{kx+ey=n}eF(k).$$

As

$$\sum_{kx+ey=n} eF(k)=\sum_{n_1+n_2=n}\sum_{e|n_1} e\sum_{d|n_2}F(d)=\sum_{n_1+n_2=n}\sigma(n_1)\sum_{d|n_2}F(d),$$

the required result follows. □

8 Proof of Theorem 1.1.

We begin by deriving a very simple identity that we shall need. Let $x\in\mathbb{R}$ be such that $|x|<1$. From the well known series

$$\sum_{n=1}^{\infty}x^n=\frac{x}{(1-x)},\quad \sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2},\quad \sum_{n=1}^{\infty}n^2x^n=\frac{x+x^2}{(1-x)^3},$$

we obtain

$$\sum_{n=1}^{\infty}(n-1)x^n=\frac{x^2}{(1-x)^2},\quad \sum_{n=1}^{\infty}n(n-1)x^n=\frac{2x^2}{(1-x)^3}.$$

Hence, for $d\in\mathbb{R}$, we have

$$\sum_{n=1}^{\infty}(n(n-1)-d(n-1))x^n=\frac{2x^2}{(1-x)^3}-\frac{dx^2}{(1-x)^2}. \tag{8.1}$$

Now let $\theta\in\mathbb{R}$. Set $F(t)=\sin\theta t$ $(t\in\mathbb{Z})$. Clearly $F(-t)=-F(t)$ so that F is an odd function. With this choice Theorem 1.2 yields

$$-4\sum_{ax+by+cz=n}\sin a\theta\sin b\theta\sin c\theta = \sum_{d|n}\left(\frac{d^2-3d+2}{2}+3\left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right)\right)\sin d\theta$$
$$+3\sum_{d|n}\sum_{1\le k<d}\left(2d-2\frac{n}{d}-k\right)\sin k\theta$$
$$-6\sum_{n=n_1+n_2}\sigma(n_1)\sum_{d|n_2}\sin d\theta.$$

Multiplying both sides by q^n, and summing over $n = 1, 2, 3, \ldots$, we obtain

$$\begin{aligned} -4\left(\sum_{a,x=1}^{\infty} \sin a\theta q^{ax}\right)^3 &= \sum_{n=1}^{\infty} q^n \sum_{d\mid n} \frac{d^2-3d+2}{2} \sin d\theta \\ &+3\sum_{n=1}^{\infty} q^n \sum_{d\mid n} \left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right) \sin d\theta \\ &+3\sum_{n=1}^{\infty} q^n \sum_{d\mid n} \sum_{1\le k<d} \left(2d-2\frac{n}{d}-k\right) \sin k\theta \\ &-6\sum_{n=1}^{\infty} q^n \sum_{n=n_1+n_2} \sigma(n_1) \sum_{d\mid n_2} \sin d\theta. \end{aligned}$$

First we observe that

$$\sum_{a,x=1}^{\infty} \sin a\theta q^{ax} = \sum_{a=1}^{\infty} \sin a\theta \frac{q^a}{1-q^a}.$$

Secondly we have

$$\begin{aligned} \sum_{n=1}^{\infty} q^n \sum_{d\mid n} \frac{d^2-3d+2}{2} \sin d\theta &= \sum_{d,e=1}^{\infty} q^{de} \frac{d^2-3d+2}{2} \sin d\theta \\ &= \sum_{d=1}^{\infty} \frac{d^2-3d+2}{2} \sin d\theta \frac{q^d}{1-q^d}. \end{aligned}$$

Thirdly, appealing to (8.1) with $x = q^d$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} q^n \sum_{d\mid n} \left(\frac{n}{d}-1\right)\left(\frac{n}{d}-d\right) \sin d\theta &= \sum_{d,e=1}^{\infty} q^{de}(e-1)(e-d) \sin d\theta \\ &= \sum_{d=1}^{\infty} \sin d\theta \left(\sum_{e=1}^{\infty} (e(e-1)-d(e-1))q^{de}\right) \\ &= \sum_{d=1}^{\infty} \sin d\theta \left(\frac{2q^{2d}}{(1-q^d)^3} - \frac{dq^{2d}}{(1-q^d)^2}\right). \end{aligned}$$

Fourthly we have

$$\sum_{n=1}^{\infty} q^n \sum_{d\mid n} \sum_{1\le k<d} \left(2d-2\frac{n}{d}-k\right) \sin k\theta$$

$$= \sum_{d,e=1}^{\infty} q^{de} \left(2d \sum_{k=1}^{d-1} \sin k\theta - 2e \sum_{k=1}^{d-1} \sin k\theta - \sum_{k=1}^{d-1} k \sin k\theta \right)$$

$$= 2\sum_{d=1}^{\infty} \frac{dq^d}{(1-q^d)} \sum_{k=1}^{d-1} \sin k\theta - 2\sum_{d=1}^{\infty} \frac{q^d}{(1-q^d)^2} \sum_{k=1}^{d-1} \sin k\theta - \sum_{d=1}^{\infty} \frac{q^d}{(1-q^d)} \sum_{k=1}^{d-1} k \sin k\theta$$

$$= \sum_{d=1}^{\infty} \sum_{k=1}^{d-1} \left(2d - \frac{2}{1-q^d} - k \right) \sin k\theta \frac{q^d}{1-q^d}.$$

Fifthly we have

$$\begin{aligned} & \sum_{n=1}^{\infty} q^n \sum_{n=n_1+n_2} \sigma(n_1) \sum_{d|n_2} \sin d\theta \\ &= \sum_{n_1,n_2=1}^{\infty} q^{n_1+n_2} \sigma(n_1) \sum_{d|n_2} \sin d\theta \\ &= \left(\sum_{n_1=1}^{\infty} q^{n_1} \sum_{e|n_1} e \right) \left(\sum_{n_2=1}^{\infty} q^{n_2} \sum_{d|n_2} \sin d\theta \right) \\ &= \left(\sum_{e,f=1}^{\infty} e q^{ef} \right) \left(\sum_{c,d=1}^{\infty} \sin d\theta q^{cd} \right) \\ &= \left(\sum_{e=1}^{\infty} \frac{eq^e}{1-q^e} \right) \left(\sum_{d=1}^{\infty} \sin d\theta \frac{q^d}{1-q^d} \right). \end{aligned}$$

Putting these results together, we obtain

$$\begin{aligned} & -4 \left(\sum_{n=1}^{\infty} \sin n\theta \frac{q^n}{1-q^n} \right)^3 \\ &= \sum_{n=1}^{\infty} \left(\frac{n^2-3n+2}{2} + 3 \left(\frac{2q^n}{(1-q^n)^2} - \frac{nq^n}{1-q^n} \right) \right) \sin n\theta \frac{q^n}{1-q^n} \\ &\quad +3 \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \left(2n - \frac{2}{1-q^n} - k \right) \sin k\theta \frac{q^n}{1-q^n} \\ &\quad -6 \left(\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right) \left(\sum_{n=1}^{\infty} \sin n\theta \frac{q^n}{1-q^n} \right), \end{aligned}$$

from which Theorem 1.1 follows.

We close by remaking that Liu [2, Theorem 11, p. 148] has obtained an analogous formula for

$$\left(\sum_{n=0}^{\infty} \sin(2n+1)\theta \frac{q^{2n+1}}{1-q^{4n+2}}\right)^3$$

by means of elliptic functions.

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