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SUMS OF SIXTEEN AND TWENTY-FOUR TRIANGULAR NUMBERS

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ABSTRACT. The triangular numbers are the integers $m(m+1)/2$, $m = 0, 1, 2, \dots$. For a positive integer k, we let $\delta_k(n)$ denote the number of representations of the nonnegative integer n as the sum of k triangular numbers. In 1994, using advanced methods, Kac and Wakimoto gave formulae for $\delta_{16}(n)$ and $\delta_{24}(n)$. Using a recent elementary identity due to Huard, Ou, Spearman and Williams, elementary proofs are given of these formulae.

1. Introduction. Let $\mathbf{N}$ denote the set of natural numbers. For $k \in \mathbf{N}$ and $n \in \mathbf{N} \cup \{0\}$ we let $\delta_k(n)$ denote the number of representations of n as the sum of k triangular numbers so that $\delta_k(0) = 1$ and $\delta_k(1) = k$. Arithmetic formulae for $\delta_2(n)$, $\delta_4(n)$, $\delta_6(n)$ and $\delta_8(n)$ are classical and well known. Elementary proofs of these formulae have been given, see Huard, Ou, Spearman and Williams [**2**]. Kac and Wakimoto [**3**, p. 452] using the representation theory of affine super-algebras have shown that

$$\delta_{16}(n) = \frac{1}{192} \sum_{\substack{a,b,x,y \in \mathbf{N} \\ ax+by=2n+4 \\ a \equiv b \equiv x \equiv y \equiv 1 \pmod 2 \\ a>b}} ab(a^2-b^2)^2 \tag{1.1}$$

and

$$\delta_{24}(n) = \frac{1}{72} \sum_{\substack{a,b,x,y \in \mathbf{N} \\ ax+by=n+3 \\ x \equiv y \equiv 1 \pmod 2 \\ a>b}} a^3b^3(a^2-b^2)^2. \tag{1.2}$$

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We show that these formulae can be proved by entirely elementary means by making use of the following elementary identity proved recently by Huard, Ou, Spearman and Williams [**2**]. This identity is a generalization of an identity of Liouville.

Theorem. *Let* $f : \mathbf{Z}^4 \to \mathbf{C}$ *be such that*

$$f(a,b,x,y) - f(x,y,a,b) = f(-a,-b,x,y) - f(x,y,-a,-b)$$

for each $(a,b,x,y) \in \mathbf{Z}^4$. *Then, for each* $n \in \mathbf{N}$, *we have*

$$\begin{aligned}\sum_{ax+by=n} &\big(f(a,b,x,-y) - f(a,-b,x,y) + f(a,a-b,x+y,y)\\ &- f(a,a+b,y-x,y) + f(b-a,b,x,x+y) - f(a+b,b,x,x-y)\big)\\ = \sum_{d\,|\,n}\sum_{x<d} &\big(f(0,n/d,x,d) + f(n/d,0,d,x) + f(n/d,n/d,d-x,-x)\\ &- f(x,x-d,n/d,n/d) - f(x,d,0,n/d) - f(d,x,n/d,0)\big),\end{aligned}$$

where the sum on the lefthand side of the identity is over all $(a,b,x,y) \in \mathbf{N}^4$ *satisfying* $ax+by=n$, *the inner sum on the righthand side is over all positive integers* x *satisfying* $x<d$, *and the outer sum on the righthand side is over all positive integers* d *dividing* n.

The proof of this identity involves only the manipulation of finite sums.

2. The sums $S_{e,f}(n)$**.** For $m \in \mathbf{N}$ and $n \in \mathbf{N}$ let $\sigma_m(n)$ denote the sum of the mth powers of the positive divisors of n. We set $\sigma(n) = \sigma_1(n)$. If $l \notin \mathbf{N}$ we set $\sigma_m(l) = 0$. For $e \in \mathbf{N}$ and $f \in \mathbf{N}$ we define

$$S_{e,f}(n) := \sum_{m=1}^{n-1} \sigma_e(m)\sigma_f(n-m). \tag{2.1}$$

Clearly

$$S_{e,f}(n) = \sum_{\substack{a,b,x,y\in\mathbf{N}\\ ax+by=n}} a^e b^f = S_{f,e}(n). \tag{2.2}$$

From this point on we write $\sum_{ax+by=n}$ for $\sum_{\substack{a,b,x,y\in\mathbf{N}\\ ax+by=n}}$.

The sums $S_{e,f}(n)$ can be evaluated explicitly in an elementary manner for $e \in \mathbf{N}$ and $f \in \mathbf{N}$ satisfying

$$e \equiv f \equiv 1 \pmod 2, \quad e+f = 2,\ 4,\ 6,\ 8,\ 12,$$

by taking particular choices of $f(a,b,x,y)$ in the Theorem, see [**2**]. In Section 5 we need the values of $S_{1,5}(n)$ and $S_{3,3}(n)$. The formula

$$S_{1,5}(n) = \frac{1}{504}\,(20\sigma_7(n) + (21-42n)\sigma_5(n) + \sigma(n)) \tag{2.3}$$

is due to Ramanujan [**5**, Table IV] and the formula

$$S_{3,3}(n) = \frac{1}{120}\,(\sigma_7(n) - \sigma_3(n)) \tag{2.4}$$

to Glaisher [**1**, p. 35]. Formulae (2.3) and (2.4) follow from the Theorem by choosing $f(a,b,x,y) = xy^5 + x^5y - 20x^3y^3$ and $f(a,b,x,y) = xy^5 + x^5y - 2x^3y^3$, respectively, see [**2**].

The evaluation of the sums $S_{1,9}(n)$, $S_{3,7}(n)$ and $S_{5,5}(n)$ requires the Ramanujan tau function $\tau(n)$ and so cannot be considered elementary. However, for our purposes, we do not require the evaluation of each of these sums individually. We only need the linear combination $4S_{3,7}(n)+5S_{5,5}(n)$. We evaluate this linear combination and the related linear combination $25S_{1,9}(n) + 48S_{3,7}(n)$ in an elementary way from the Theorem. These evaluations can also be obtained from the work of Lahiri [**4**, p. 34].

Corollary 1. *For $n \in \mathbf{N}$ we have*

(a) $$25S_{1,9}(n) + 48S_{3,7}(n) = \frac{91}{220}\,\sigma_{11}(n) + \left(\frac{25}{24} - \frac{5}{4}\,n\right)\sigma_9(n) - \frac{1}{5}\,\sigma_7(n) - \frac{1}{10}\,\sigma_3(n) + \frac{25}{264}\,\sigma(n)$$

and

(b) $$4S_{3,7}(n) + 5S_{5,5}(n) = \frac{13}{2520}\,\sigma_{11}(n) - \frac{1}{60}\,\sigma_7(n) + \frac{5}{252}\,\sigma_5(n) - \frac{1}{120}\,\sigma_3(n).$$

Proof. (a) Choosing

$$f(a,b,x,y) = a^3b^7 - 21a^5b^5$$

in the Theorem we obtain

$$\sum_{ax+by=n} \big(196a^9b + 350a^7b^3 + 418a^3b^7 + 204ab^9\big)$$
$$= -20\sum_{d\mid n}\left(\frac{n}{d}\right)^{10}(d-1) + \sum_{d\mid n}\sum_{x<d}\big(20x^{10} - 98x^9d + 189x^8d^2 - 176x^7d^3 + 70x^6d^4 + 42x^5d^5 - 7x^4d^6\big),$$

so that after some calculation

$$400S_{1,9}(n) + 768S_{3,7}(n) = \frac{364}{55}\,\sigma_{11}(n) + \left(\frac{50}{3} - 20n\right)\sigma_9(n) - \frac{16}{5}\,\sigma_7(n) - \frac{8}{5}\,\sigma_3(n) + \frac{50}{33}\,\sigma(n).$$

Dividing both sides by 16, we obtain (a).

(b) Choosing

$$f(a,b,x,y) = a^3b^7 - a^5b^5$$

in the Theorem we obtain

$$\sum_{ax+by=n} \big(-4a^9b - 50a^7b^3 - 40a^5b^5 + 18a^3b^7 + 4ab^9\big)$$
$$= \sum_{d\mid n}\sum_{x<d}\big(2x^9d - 11x^8d^2 + 24x^7d^3 - 30x^6d^4 + 22x^5d^5 - 7x^4d^6\big),$$

so that after some calculation

$$-32S_{3,7}(n) - 40S_{5,5}(n) = -\frac{13}{315}\,\sigma_{11}(n) + \frac{2}{15}\,\sigma_7(n) - \frac{10}{63}\,\sigma_5(n) + \frac{1}{15}\,\sigma_3(n).$$

Dividing both sides by -8, we obtain (b).

3. The sums $A_{e,f}(n)$. For e, f, $n \in \mathbf{N}$ we define

$$A_{e,f}(n) := \sum_{m<n/2} \sigma_e(m)\sigma_f(n-2m), \tag{3.1}$$

where m runs through the positive integers satisfying $m < n/2$. We note that

$$A_{e,f}(n) = \sum_{2ax+by=n} a^e b^f. \tag{3.2}$$

In [**2**, Theorem 15] the formulae

$$\begin{aligned} 3A_{1,5}(n) + 8A_{3,3}(n) = \frac{1}{840}\big(&28\sigma_7(n) + (105-105n)\sigma_5(n) - 28\sigma_3(n) \\ &+ 128\sigma_7(n/2) - 28\sigma_3(n/2) + 5\sigma(n/2)\big) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} 2A_{3,3}(n) + 3A_{5,1}(n) = \frac{1}{840}\big(&2\sigma_7(n) - 7\sigma_3(n) + 5\sigma(n) + 112\sigma_7(n/2) \\ &+ (105-210n)\sigma_5(n/2) - 7\sigma_3(n/2)\big) \end{aligned} \tag{3.4}$$

are deduced from the Theorem by choosing

$$f(a,b,x,y) = (-ab^5 + 10a^3b^3 - 12a^4b^2)F_2(x)$$

and

$$f(a,b,x,y) = (ab^5 - 10a^3b^3 + 12a^4b^2 - 36a^6)F_2(x),$$

respectively, where for $x \in \mathbf{Z}$

$$F_2(x) = \begin{cases} 1 & \text{if } 2 \mid x, \\ 0 & \text{if } 2 \nmid x. \end{cases}$$

The next result is an elementary consequence of the Theorem. It relates $A_{3,7}(n)$, $A_{5,5}(n)$ and $A_{7,3}(n)$, and is needed in Section 6.

Corollary 2. *For* $n \in \mathbf{N}$

$$\begin{aligned} 2S_{3,7}(n) - 5S_{5,5}(n) + 8S_{3,7}(n/2) - 5S_{5,5}(n/2) + 3S_{3,3}(n) - 3S_{3,3}(n/2) \\ + 4A_{3,7}(n) + 10A_{5,5}(n) - 14A_{7,3}(n) = 0. \end{aligned}$$

Proof. Choosing

$$f(a,b,x,y) = \left(-\frac{7}{9}\,a^3b^7 + \frac{4}{3}\,a^5b^5 - \frac{1}{3}\,a^7b^3 - \frac{2}{9}\,a^9b\right) F_2(x)$$

in the Theorem, we obtain

$$\begin{aligned} 2 \sum_{2ax+by=n} &(34a^3b^7 + 70a^5b^5 + 16a^7b^3) \\ &+ \sum_{\substack{ax+by=n \\ x\equiv y \pmod 2}} \left(\frac{14}{9}\,a^3b^7 + 30\,a^5b^5 + \frac{256}{9}\,a^7b^3\right) \\ &= \sum_{\substack{d\mid n \\ 2\mid n/d}} \sum_{x<d} \left(4x^8d^2 - \frac{121}{9}\,x^7d^3 + \frac{185}{9}\,x^6d^4 - \frac{49}{3}\,x^5d^5\right. \\ &\qquad \left. + \frac{49}{9}\,x^4d^6 - \frac{4}{9}\,x^3d^7 + \frac{2}{9}\,xd^9\right) \\ &\quad + \sum_{d\mid n} \sum_{x<d} \left(\frac{2}{9}\,x^9d + \frac{1}{3}\,x^7d^3 - \frac{4}{3}\,x^5d^5 + \frac{7}{9}\,x^3d^7\right). \end{aligned}$$

Using the inclusion-exclusion principle on the second sum on the lefthand side, we obtain after some calculation

$$\begin{aligned} 30(S_{3,7}(n) &+ 2S_{3,7}(n/2)) + 30(S_{5,5}(n) + 2S_{5,5}(n/2)) \\ &+ (4A_{3,7}(n) + 10A_{5,5}(n) - 14A_{7,3}(n)) \\ &= \frac{169}{2520}\,\sigma_{11}(n/2) - \frac{23}{120}\,\sigma_7(n/2) + \frac{65}{252}\,\sigma_5(n/2) - \frac{2}{15}\,\sigma_3(n/2) \\ &\quad + \frac{13}{360}\,\sigma_{11}(n) - \frac{17}{120}\,\sigma_7(n) + \frac{5}{36}\,\sigma_5(n) - \frac{1}{30}\,\sigma_3(n). \end{aligned}$$

Hence

$$\begin{aligned} 2S_{3,7}(n) &- 5S_{5,5}(n) + 8S_{3,7}(n/2) - 5S_{5,5}(n/2) + 3S_{3,3}(n) - 3S_{3,3}(n/2) \\ &+ 4A_{3,7}(n) + 10A_{5,5}(n) - 14A_{7,3}(n) \\ &= -7(4S_{3,7}(n) + 5S_{5,5}(n)) - 13(4S_{3,7}(n/2) + 5S_{5,5}(n/2)) \\ &\quad + 3S_{3,3}(n) - 3S_{3,3}(n/2) + \frac{13}{360}\,\sigma_{11}(n) - \frac{17}{120}\,\sigma_7(n) \\ &\quad + \frac{5}{36}\,\sigma_5(n) - \frac{1}{30}\,\sigma_3(n) + \frac{169}{2520}\,\sigma_{11}(n/2) \\ &\quad - \frac{23}{120}\,\sigma_7(n/2) + \frac{65}{252}\,\sigma_5(n/2) - \frac{2}{15}\,\sigma_3(n/2). \end{aligned}$$

Appealing to Corollary 1(b) for $4S_{3,7}+5S_{5,5}$ and to (2.4) for $S_{3,3}$, we find that the righthand side is 0, proving the asserted result.

4. Formulae for $\delta_{16}(n)$ and $\delta_{24}(n)$. First we determine $\delta_{16}(n)$ in terms of the sums $S_{3,3}$ and $A_{3,3}$. As

$$\delta_8(m)=\sigma_3(m+1)-\sigma_3((m+1)/2),\quad \text{for all } m\in\mathbf{N}\cup\{0\}, \tag{4.1}$$

see [**2**, Theorem 12] for an elementary proof, we have

$$\begin{aligned}
\delta_{16}(n) &= \sum_{\substack{r,s\ge 0\\ r+s=n}} \delta_8(r)\delta_8(s)\\
&= \sum_{\substack{r,s\ge 0\\ r+s=n}} \big(\sigma_3(r+1)-\sigma_3((r+1)/2)\big)\big(\sigma_3(s+1)-\sigma_3((s+1)/2)\big)\\
&= \sum_{\substack{r,s\ge 1\\ r+s=n+2}} \big(\sigma_3(r)-\sigma_3(r/2)\big)\big(\sigma_3(s)-\sigma_3(s/2)\big)\\
&= \sum_{r+s=n+2}\sigma_3(r)\sigma_3(s)-\sum_{2r+s=n+2}\sigma_3(r)\sigma_3(s)\\
&\quad -\sum_{r+2s=n+2}\sigma_3(r)\sigma_3(s)+\sum_{2r+2s=n+2}\sigma_3(r)\sigma_3(s);
\end{aligned}$$

that is,

$$\delta_{16}(n)=S_{3,3}(n+2)+S_{3,3}((n+2)/2)-2A_{3,3}(n+2). \tag{4.2}$$

Next we express $\delta_{24}(n)$ in terms of the sums $S_{3,7}$, $S_{3,3}$, $A_{3,7}$ and $A_{7,3}$. Clearly

$$\delta_{24}(n)=\sum_{\substack{r,s,t\ge 0\\ r+s+t=n}}\delta_8(r)\delta_8(s)\delta_8(t),$$

which, appealing to (4.1), becomes as in the derivation of (4.2)

$$\delta_{24}(n)=U_1-3U_2+3U_3-U_4,$$

where

$$U_1 := \sum_{\substack{r,s,t \geq 1 \\ r+s+t=n+3}} \sigma_3(r)\sigma_3(s)\sigma_3(t),$$
$$U_2 := \sum_{\substack{r,s,t \geq 1 \\ 2r+s+t=n+3}} \sigma_3(r)\sigma_3(s)\sigma_3(t),$$
$$U_3 := \sum_{\substack{r,s,t \geq 1 \\ 2r+2s+t=n+3}} \sigma_3(r)\sigma_3(s)\sigma_3(t),$$
$$U_4 := \sum_{\substack{r,s,t \geq 1 \\ 2r+2s+2t=n+3}} \sigma_3(r)\sigma_3(s)\sigma_3(t).$$

We next evaluate U_3. We have

$$\begin{aligned} U_3 &= \sum_{1 \leq t < n} \sigma_3(t) S_{3,3}((n+3-t)/2) \\ &= \sum_{1 \leq u < (n+3)/2} \sigma_3(n+3-2u) S_{3,3}(u), \end{aligned}$$

as $S_{3,3}(1) = 0$. Then, appealing to (2.4), we obtain

$$U_3 = \frac{1}{120} A_{7,3}(n+3) - \frac{1}{120} A_{3,3}(n+3).$$

Similarly we find that

$$\begin{aligned} U_1 &= \frac{1}{120} S_{3,7}(n+3) - \frac{1}{120} S_{3,3}(n+3), \\ U_2 &= \frac{1}{120} A_{3,7}(n+3) - \frac{1}{120} A_{3,3}(n+3), \\ U_4 &= \frac{1}{120} S_{3,7}((n+3)/2) - \frac{1}{120} S_{3,3}((n+3)/2). \end{aligned}$$

Thus we deduce that

$$\begin{aligned} 120\delta_{24}(n) = S_{3,7}(n+3) &- S_{3,3}(n+3) \\ &- S_{3,7}((n+3)/2) + S_{3,3}((n+3)/2) \\ &- 3A_{3,7}(n+3) + 3A_{7,3}(n+3). \end{aligned} \tag{4.3}$$

5. Elementary proof of the Kac-Wakimoto formula for $\delta_{16}(n)$. We denote the sum on the righthand side of (1.1) by $E(n)$, so we wish to prove that $\delta_{16}(n) = E(n)/192$. Mapping $(a, b, x, y) \to (b, a, y, x)$ in $E(n)$ we see that

$$E(n) = \sum_{\substack{ax+by=2n+4 \\ a\equiv b\equiv x\equiv y\equiv 1 \pmod 2 \\ a>b}} ab(a^2-b^2)^2 = \sum_{\substack{ax+by=2n+4 \\ a\equiv b\equiv x\equiv y\equiv 1 \pmod 2 \\ a<b}} ab(a^2-b^2)^2.$$

Also

$$\sum_{\substack{ax+by=2n+4 \\ a\equiv b\equiv x\equiv y\equiv 1 \pmod 2 \\ a=b}} ab(a^2-b^2)^2 = 0.$$

Thus

$$E(n) = \frac{1}{2} \sum_{\substack{ax+by=2n+4 \\ a\equiv b\equiv x\equiv y\equiv 1 \pmod 2}} ab(a^2-b^2)^2.$$

If $a \equiv x \equiv 1 \pmod 2$, then the equation $ax + by = 2n + 4$ forces $b \equiv y \equiv 1 \pmod 2$ so that

$$E(n) = \frac{1}{2} \sum_{\substack{ax+by=2n+4 \\ a\equiv x\equiv 1 \pmod 2}} ab(a^2-b^2)^2.$$

By the inclusion-exclusion principle we have

$$E(n) = \frac{1}{2}\left(T_1 - T_2 - T_3 + T_4\right), \tag{5.1}$$

where

$$T_1 := \sum_{ax+by=2n+4} ab(a^2-b^2)^2, \qquad T_2 := \sum_{2ax+by=2n+4} ab(a^2-b^2)^2,$$
$$T_3 := \sum_{2ax+by=2n+4} 2ab(4a^2-b^2)^2, \quad T_4 := \sum_{4ax+by=2n+4} 2ab(4a^2-b^2)^2.$$

Expanding the squares in the expressions for the T_i, and appealing to (2.2) and (3.2), we obtain

$$\begin{aligned}
T_1 &= 2S_{1,5}(2n+4) - 2S_{3,3}(2n+4),\\
T_2 &= 36S_{1,5}(n+2) - 18S_{3,3}(n+2)\\
&\quad - 2A_{1,5}(n+2) + 16A_{3,3}(n+2) - 32A_{5,1}(n+2),\\
T_3 &= 162S_{1,5}(n+2) - 144S_{3,3}(n+2)\\
&\quad - 64A_{1,5}(n+2) + 128A_{3,3}(n+2) - 64A_{5,1}(n+2),\\
T_4 &= -128S_{1,5}((n+2)/2) + 128S_{3,3}((n+2)/2)\\
&\quad + 66A_{1,5}(n+2) - 144A_{3,3}(n+2) + 96A_{5,1}(n+2).
\end{aligned}$$

Thus

$$\begin{aligned}
E(n) = S_{1,5}(2n+4) &- S_{3,3}(2n+4) - 99S_{1,5}(n+2)\\
&+ 81S_{3,3}(n+2) - 64S_{1,5}((n+2)/2) + 64S_{3,3}((n+2)/2)\\
&+ 22\{3A_{1,5}(n+2) + 8A_{3,3}(n+2)\}\\
&+ 32\{2A_{3,3}(n+2) + 3A_{5,1}(n+2)\} - 384A_{3,3}(n+2).
\end{aligned}$$

Appealing to (2.3), (2.4), (3.3) and (3.4) for $S_{1,5}$, $S_{3,3}$, $3A_{1,5}+8A_{3,3}$ and $2A_{3,3}+3A_{5,1}$, respectively, and making use of the elementary identity

$$\sigma_k(2n) = (2^k+1)\sigma_k(n) - 2^k\sigma_k(n/2), \quad k, n \in \mathbf{N},$$

we deduce that

$$\begin{aligned}
\frac{E(n)}{192} = \frac{1}{120}\,\sigma_7(n+2) &- \frac{1}{120}\,\sigma_3(n+2) + \frac{1}{120}\,\sigma_7((n+2)/2)\\
&- \frac{1}{120}\,\sigma_3((n+2)/2) - 2A_{3,3}(n+2).
\end{aligned}$$

Then, by (2.4) and (4.2), we obtain

$$\frac{E(n)}{192} = S_{3,3}(n+2) + S_{3,3}((n+2)/2) - 2A_{3,3}(n+2) = \delta_{16}(n).$$

6. Elementary proof of the Kac-Wakimoto formula for $\delta_{24}(n)$. We consider the sum on the righthand side of (1.2). This sum is

$$G(n) := \frac{1}{2} \sum_{\substack{ax+by=n+3\\ x\equiv y\equiv 1 \pmod 2}} a^3b^3(a^2-b^2)^2,$$

so we wish to prove that $\delta_{24}(n) = G(n)/72$. We have

$$G(n) = X_1 - X_2, \tag{6.1}$$

where

$$X_1 := \sum_{\substack{ax+by=n+3 \\ x\equiv y\equiv 1 \pmod 2}} a^3 b^7, \qquad X_2 := \sum_{\substack{ax+by=n+3 \\ x\equiv y\equiv 1 \pmod 2}} a^5 b^5.$$

By the inclusion-exclusion principle we obtain

$$X_1 = S_{3,7}(n+3) - A_{3,7}(n+3) - A_{7,3}(n+3) + S_{3,7}((n+3)/2) \tag{6.2}$$

and

$$X_2 = S_{5,5}(n+3) - 2A_{5,5}(n+3) + S_{5,5}((n+3)/2). \tag{6.3}$$

Appealing to (4.3), (6.1), (6.2) and (6.3), we obtain

$$\begin{aligned} 5G(n) - 360\delta_{24}(n) = {} & 2S_{3,7}(n+3) - 5S_{5,5}(n+3) + 8S_{3,7}((n+3)/2) \\ & - 5S_{5,5}((n+3)/2) + 3S_{3,3}(n+3) \\ & - 3S_{3,3}\big((n+3)/2) + 4A_{3,7}(n+3) \\ & + 10A_{5,5}(n+3) - 14A_{7,3}(n+3)\big), \end{aligned}$$

and the righthand side is 0 by Corollary 2, as desired.

7. Conclusion. It would be interesting to know if the Kac-Wakimoto formulae for $\delta_{4k^2}(n)$ and $\delta_{4k(k+1)}(n)$, $k \in \mathbf{N}$, can be proved by entirely elementary means.

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