Far East J. Math. Sci. (FJMS) 11(2) (2003), 233-240

$n = \Delta + \Delta + 2(\Delta + \Delta)$

KENNETH S. WILLIAMS

(Received June 8, 2003)

Submitted by K. K. Azad

Abstract

Let n be a nonnegative integer. An explicit formula is given for the number of quadruples (t_1, t_2, t_3, t_4) of triangular numbers such that

$$n = t_1 + t_2 + 2(t_3 + t_4).$$

As a consequence of this formula we deduce that every nonnegative integer is of the form $t_1 + t_2 + 2(t_3 + t_4)$ for some triangular numbers t_1, t_2, t_3, t_4.

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. The triangular numbers are the nonnegative integers

$$T_k = \frac{1}{2} k(k+1), \quad k \in \mathbb{N}_0,$$

so that

$$T_0 = 0, T_1 = 1, T_2 = 3, T_3 = 6, T_4 = 10, T_5 = 15, \ldots.$$

2000 Mathematics Subject Classification: Primary 11E25.

Key words and phrases: triangular numbers.

The author was supported by Natural Sciences and Engineering Research Council of Canada grant A-7233.

We set

$$\Delta = \{T_k \mid k \in \mathbb{N}_0\}.$$

For $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ we let

$$\delta_m(n) = \text{card}\{(t_1, \ldots, t_m) \in \Delta^m \mid n = t_1 + \cdots + t_m\},$$

so that $\delta_m(n)$ counts the number of representations of n as the sum of m triangular numbers. It is an easily proved classical result that

$$\delta_2(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid 4n+1}} \left(\frac{-4}{d}\right), \quad n \in \mathbb{N}_0, \tag{1.1}$$

where d runs through the positive integers dividing $4n + 1$ and

$$\left(\frac{-4}{d}\right) = \begin{cases} +1, & \text{if } d \equiv 1 \pmod 4, \\ 0, & \text{if } d \equiv 0 \pmod 2, \\ -1, & \text{if } d \equiv -1 \pmod 4, \end{cases}$$

see for example [4, pp. 77-78]. Similarly

$$\delta_4(n) = \sigma(2n + 1), \quad n \in \mathbb{N}_0, \tag{1.2}$$

where

$$\sigma(m) = \begin{cases} \sum\limits_{\substack{d \in \mathbb{N} \\ d \mid m}} d, & m \in \mathbb{N}, \\ 0, & m \notin \mathbb{N}. \end{cases}$$

The result (1.2) was known to Legendre [2]. A proof using modular forms has been given by Ono, Robins and Wahl [4, pp. 79-80]. An elementary arithmetic proof has been given by Huard, Ou, Spearman and Williams [1, pp. 259-262].

In this paper, we determine

$$R(n) = \text{card}\{(t_1, t_2, t_3, t_4) \in \Delta^4 \mid n = t_1 + t_2 + 2(t_3 + t_4)\} \tag{1.3}$$

by entirely arithmetic means. We make use of the elementary result (1.1) as well as the recent elementary identity due to Huard, Ou, Spearman and Williams [1, Theorem 1, p. 230], which was used to prove (1.2). This identity is given in Section 2 as Proposition 1. In Section 3, we prove the following result.

Theorem. *For* $n \in \mathbb{N}_0$ *we have*

$$R(n) = \frac{1}{4} \sum_{\substack{d \in \mathbb{N} \\ d \mid 4n+3}} \left(d - (-1)^{(d-1)/2}\right).$$

An immediate consequence of this theorem is that every nonnegative integer is of the form $t_1 + t_2 + 2t_3 + 2t_4$ for triangular numbers t_1, t_2, t_3, t_4.

2. Preliminary Results

The elementary identity of Huard, Ou, Spearman and Williams [1, Theorem 1, p. 230] mentioned in Section 1 is the following result:

Proposition 1. *Let* $f : \mathbb{Z}^4 \mapsto \mathbb{C}$ *be such that*

$$f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all $(a, b, x, y) \in \mathbb{Z}^4$. *Then, for* $n \in \mathbb{N}$, *we have*

$$\sum_{\substack{(a, b, x, y) \in \mathbb{N}^4 \\ ax + by = n}} (f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y)$$

$$- f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y))$$

$$= \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \sum_{x=1}^{d-1} (f(0, n/d, x, d) + f(n/d, 0, d, x) + f(n/d, n/d, d-x, -x)$$

$$- f(x, x-d, n/d, n/d) - f(x, d, 0, n/d) - f(d, x, n/d, 0)).$$

For $x \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $k \geq 2$, we let

$$F_k(x) = \begin{cases} 1, & \text{if } k \mid x, \\ 0, & \text{if } k \nmid x. \end{cases}$$

As usual we set

$$d(m) = \begin{cases} \sum_{\substack{d \in \mathbb{N} \\ d \mid m}} 1, & \text{if } m \in \mathbb{N}, \\ 0, & \text{if } m \notin \mathbb{N}. \end{cases}$$

Taking $f(a, b, x, y) = f(a)F_2(x)$ in Proposition 1, where $f : \mathbb{Z} \mapsto \mathbb{C}$ is an even function, we obtain the following interesting identity.

Proposition 2. *Let* $f : \mathbb{Z} \mapsto \mathbb{C}$ *be an even function. Then, for* $n \in \mathbb{N}$, *we have*

$$\sum_{\substack{(a, b, x, y) \in \mathbb{N}^4 \\ 2ax + by = n}} (f(a - b) - f(a + b))$$

$$= \frac{1}{2} f(0)(\sigma(n) - d(n) - d(n/2)) + \frac{1}{2} \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(1 + \frac{n}{d}\right) f(d)$$

$$+ \frac{1}{2} \sum_{\substack{d \in \mathbb{N} \\ d \mid \frac{n}{2}}} \left(1 - 2d + \frac{2n}{d}\right) f(d) - \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\sum_{v=1}^{d} f(v) \right) - \sum_{\substack{d \in \mathbb{N} \\ d \mid \frac{n}{2}}} \left(\sum_{v=1}^{d} f(v) \right).$$

Proposition 2 is similar to the following identity of Liouville [3, p. 284].

Proposition 3. *Let* $f : \mathbb{Z} \mapsto \mathbb{C}$ *be an even function. Then, for* $n \in \mathbb{N}$,

$$\sum_{\substack{(a, b, x, y) \in \mathbb{N}^4 \\ ax + by = n}} (f(a - b) - f(a + b))$$

$$= f(0)(\sigma(n) - d(n)) + \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(1 - d + \frac{2n}{d}\right) f(d) - 2 \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} \left(\sum_{v=1}^{d} f(v) \right).$$

Taking $f(x) = F_4(x)$ in Proposition 2, and replacing n by $4n + 3$, we obtain, as

$$F_4(a - b) - F_4(a + b) = \left(\frac{-4}{ab}\right), \quad \text{for all } a, b \in \mathbb{N},$$

the following result.

Proposition 4. *For* $n \in \mathbb{N}_0$ *we have*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ 2ax+by=4n+3}} \left(\frac{-4}{ab}\right) = \frac{1}{4} \sum_{\substack{d\in\mathbb{N} \\ d\mid 4n+3}} \left(d - \left(\frac{-4}{d}\right)\right).$$

3. Proof of Theorem

Let $n \in \mathbb{N}_0$. By (1.1) and (1.3) we have

$$R(n) = \sum_{\substack{m\in\mathbb{N}_0 \\ m\le n/2}} \delta_2(m)\delta_2(n-2m). \tag{3.1}$$

Appealing to (1.1) and (3.1), we obtain

$$R(n) = \sum_{\substack{m\in\mathbb{N}_0 \\ m\le n/2}} \left(\sum_{\substack{a\in\mathbb{N} \\ a\mid 4m+1}} \left(\frac{-4}{a}\right) \right) \left(\sum_{\substack{b\in\mathbb{N} \\ b\mid 4(n-2m)+1}} \left(\frac{-4}{b}\right) \right). \tag{3.2}$$

Now

$$\left(\frac{-4}{a}\right)\left(\frac{-4}{b}\right) = \left(\frac{-4}{ab}\right), \quad \text{for all } a, b \in \mathbb{N},$$

and

$$a \mid 4m+1,\ b \mid 4(n-2m)+1 \text{ for some } m \in \mathbb{N}_0 \text{ with } m \le n/2$$

$$\Leftrightarrow 4n+3 = 2ax+by,\ ax \equiv 1 \pmod 4 \text{ for some } x, y \in \mathbb{N}.$$

Hence (3.2) becomes

$$R(n) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ 2ax+by=4n+3 \\ ax\equiv 1 \pmod 4}} \left(\frac{-4}{ab}\right). \tag{3.3}$$

Next, we show that

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ 2ax+by=4n+3 \\ ax\equiv 3 \pmod 4}} \left(\frac{-4}{ab}\right) = 0. \tag{3.4}$$

Interchanging the roles of a and x in the sum (3.4), and noting that

$$\left(\frac{-4}{xb}\right) = \left(\frac{-4}{a^2xb}\right) = \left(\frac{-4}{ax}\right)\left(\frac{-4}{ab}\right) = \left(\frac{-4}{3}\right)\left(\frac{-4}{ab}\right) = -\left(\frac{-4}{ab}\right),$$

we obtain

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ ax\equiv 3 \pmod 4}} \left(\frac{-4}{ab}\right) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ ax\equiv 3 \pmod 4}} \left(\frac{-4}{xb}\right) = -\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ ax\equiv 3 \pmod 4}} \left(\frac{-4}{ab}\right),$$

from which (3.4) follows. Adding (3.4) to (3.1), we obtain

$$R(n) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ ax\equiv 1 \pmod 2}} \left(\frac{-4}{ab}\right). \tag{3.5}$$

If a is even, then $\left(\frac{-4}{ab}\right) = 0$ so

$$R(n) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ x\equiv 1 \pmod 2}} \left(\frac{-4}{ab}\right). \tag{3.6}$$

Next, we show that

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ x\equiv 0 \pmod 2}} \left(\frac{-4}{ab}\right) = 0. \tag{3.7}$$

Interchanging the roles of b and y in the sum in (3.7), and noting that

$$\left(\frac{-4}{ay}\right) = \left(\frac{-4}{b^2ay}\right) = \left(\frac{-4}{by}\right)\left(\frac{-4}{ab}\right) = \left(\frac{-4}{3}\right)\left(\frac{-4}{ab}\right) = -\left(\frac{-4}{ab}\right),$$

we obtain

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ x\equiv 0 \pmod 2}} \left(\frac{-4}{ab}\right) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ x\equiv 0 \pmod 2}} \left(\frac{-4}{ay}\right) = -\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=4n+3\\ x\equiv 0 \pmod 2}} \left(\frac{-4}{ab}\right),$$

from which (3.7) follows. Adding (3.6) and (3.7), we obtain

$$R(n) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ 2ax+by=4n+3}} \left(\frac{-4}{ab}\right). \tag{3.8}$$

Appealing to Proposition 4, (3.8) yields

$$R(n) = \frac{1}{4} \sum_{\substack{d\in\mathbb{N} \\ d\mid 4n+3}} \left(d - \left(\frac{-4}{d}\right)\right).$$

The theorem now follows as $\left(\frac{-4}{d}\right) = (-1)^{(d-1)/2}$ for d odd.

For odd $d \in \mathbb{N}$ we have

$$d - (-1)^{(d-1)/2} \geq d - 1 \geq 0.$$

Hence for $n \in \mathbb{N}_0$ we deduce that

$$R(n) = \frac{1}{4} \sum_{\substack{d\in\mathbb{N} \\ d\mid 4n+3 \\ d<4n+3}} (d - (-1)^{(d-1)/2}) + n + 1 \geq 0 + 0 + 1 = 1.$$

This shows that every nonnegative integer is of the form $t_1 + t_2 + 2(t_3 + t_4)$ for some triangular numbers t_1, t_2, t_3, t_4.

For example, with $n = 6$ we have

$$R(6) = \frac{1}{4} \sum_{\substack{d\in\mathbb{N} \\ d\mid 27}} (d - (-1)^{(d-1)/2}) = \frac{1}{4}(0 + 4 + 8 + 28) = 10.$$

The 10 representations $(t_1, t_2, t_3, t_4) \in \Delta^4$ in $6 = t_1 + t_2 + 2(t_3 + t_4)$ are

$(t_1, t_2, t_3, t_4) = (0, 0, 0, 3), (0, 0, 3, 0), (0, 6, 0, 0), (1, 1, 1, 1), (1, 3, 0, 1),$

$(1, 3, 1, 0), (3, 1, 0, 1), (3, 1, 1, 0), (3, 3, 0, 0), (6, 0, 0, 0).$

References

[1] J. G. Huard, Z. M. Ou, B. K. Spearman and K. S. Williams, Number Theory for the Millennium II, M. A. Bennett et al., eds., A. K. Peters Ltd., Natick, Massachusetts, 2002.

[2] A. Legendre, Traité des Fonctions Elliptiques, Vol. 3, Paris, 1828.

[3] J. Liouville, Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres (cinquième article), J. Math. Pures Appl. 3 (1858), 273-288.

[4] K. Ono, S. Robins and P. T. Wahl, On the representation of integers as sums of triangular numbers, Aequationes Math. 50 (1995), 73-94.

Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario, Canada K1S 5B6
e-mail: williams@math.carleton.ca