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THE SIMPLEST ARITHMETIC PROOF OF JACOBI'S FOUR SQUARES THEOREM

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Abstract

A simple arithmetic proof is given of Jacobi's formula for the number of representations of a positive integer as the sum of four squares.

1. Introduction

Lagrange showed in 1770 that every positive integer n is the sum of four perfect squares. Jacobi [8, 9, 10] showed that the number $r_4(n)$ of representations of n as the sum of four squares is 8 times the sum of the divisors of n that are not multiples of 4, that is,

$$r_4(n) = 8(\sigma(n) - 4\sigma(n/4)), \tag{1}$$

where $\sigma(n) = \sum_{d|n} d$ and $\sigma(n/4)$ is understood to be zero if $n/4$ is not an integer. Many proofs of (1) have been given, see for example [1], [2, p. 348], [3, p. 15], [5], [6, p. 314], [7], [12, p. 450]. Many proofs of Jacobi's formula (1) make use of the identity

$$\left(\sum_{n=-\infty}^{\infty} x^{n^2}\right)^4 = 1 + 8\sum_{n=1}^{\infty} \frac{x^n}{(1+(-x)^n)^2}, \quad |x| < 1,$$

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or a variant of it, since the coefficient of x^n on the left hand side is $r_4(n)$.

We present here what we believe to be the simplest arithmetic proof of (1). Our starting point is the formula for the number $r_2(n)$ of representations of n as the sum of two perfect squares, namely

$$r_2(n) = 4 \sum_{\substack{d|n \\ d \equiv 1(2)}} (-1)^{(d-1)/2}, \quad n \geq 1, \tag{2}$$

where we have written $d \equiv 1(2)$ for $d \equiv 1(\text{mod } 2)$. The formula (2) can be proved in an entirely elementary manner, see for example [4, p. 75], [11, p. 163].

We use the following notation

$$\tau(n) = \sum_{d|n} 1, \quad \tau_i = \tau_{i,\,4}(n) = \sum_{\substack{d|n \\ d \equiv i (\text{mod } 4)}} 1 \quad (i = 0, 1, 2, 3),$$

so that

$$\tau_0 + \tau_1 + \tau_2 + \tau_3 = \tau(n),\ \tau_0 = \tau(n/4),$$

$$\tau_2 = \tau(n/2) - \tau(n/4),\ r_2(n) = 4(\tau_1 - \tau_3). \tag{3}$$

For a real number x, we set

$$[x]^* = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - 1, & \text{if } x \text{ is an integer.} \end{cases}$$

2. Proof of (1)

Clearly,

$$r_4(n) = \sum_{k=0}^{n} r_2(k)\, r_2(n-k). \tag{4}$$

Since $r_2(0) = 1$, we can rewrite (4) as

$$r_4(n) - 2r_2(n) = \sum_{k=1}^{n-1} r_2(k)\, r_2(n-k). \tag{5}$$

Appealing to (2), (5) becomes

$$r_4(n) - 2r_2(n) = 16 \sum_{k=1}^{n-1} \sum_{\substack{a|k \\ a\equiv 1(2)}} (-1)^{(a-1)/2} \sum_{\substack{b|n-k \\ b\equiv 1(2)}} (-1)^{(b-1)/2}. \tag{6}$$

Setting $k = aA$ and $n - k = bB$ in (6), and noting that $(a-1)/2 + (b-1)/2 \equiv (a-b)/2 \pmod 2$, we obtain

$$\frac{1}{16}(r_4(n) - 2r_2(n)) = \sum_{\substack{aA+bB=n \\ a\equiv b\equiv 1(2)}} (-1)^{(a-b)/2}, \tag{7}$$

where the sum is over all positive integers a, b, A, B satisfying the stated conditions. Equation (7) can be rewritten as

$$\frac{1}{16}(r_4(n) - 2r_2(n)) = \sum_{\substack{aA+bB=n \\ a\equiv b\equiv 1(2) \\ a\equiv b(4)}} 1 - \sum_{\substack{aA+bB=n \\ a\equiv b\equiv 1(2) \\ a\equiv -b(4)}} 1. \tag{8}$$

As

$$\sum_{\substack{aA+bB=n \\ a\equiv b\equiv 1(2) \\ a\equiv \pm b(4)}} 1 = \sum_{\substack{aA+bB=n \\ a\equiv \pm b(4)}} 1 - \sum_{\substack{aA+bB=n \\ a\equiv b\equiv 0(2) \\ a\equiv \pm b(4)}} 1 = \sum_{\substack{aA+bB=n \\ a\equiv \pm b(4)}} 1 - \sum_{\substack{aA+bB=n \\ a\equiv b\equiv 0(2) \\ a\equiv b(4)}} 1,$$

equation (8) can be written as

$$\frac{1}{16}(r_4(n) - 2r_2(n)) = \sum_{\substack{aA+bB=n \\ a\equiv b(4)}} 1 - \sum_{\substack{aA+bB=n \\ a\equiv -b(4)}} 1. \tag{9}$$

It is now convenient to define

$$A_1 = \sum_{\substack{4aA+bB=n \\ A<B}} 1,\ A_2 = \sum_{\substack{4aA+bB=n \\ A>B}} 1,\ A_3 = \sum_{\substack{4aA+bB=n \\ A=B}} 1, \tag{10}$$

$$B_1 = \sum_{\substack{4aA+bB=n \\ 4a>b}} 1,\ B_2 = \sum_{\substack{4aA+bB=n \\ 4a<b}} 1,\ B_3 = \sum_{\substack{4aA+bB=n \\ 4a=b}} 1. \tag{11}$$

Then

$$\sum_{\substack{aA+bB=n\\ a\equiv b(4)}} 1 = \sum_{\substack{aA+bB=n\\ a>b\\ a\equiv b(4)}} 1 + \sum_{\substack{aA+bB=n\\ a<b\\ a\equiv b(4)}} 1 + \sum_{\substack{aA+bB=n\\ a=b\\ a\equiv b(4)}} 1$$

$$= 2\sum_{\substack{aA+bB=n\\ a>b\\ a\equiv b(4)}} 1 + \sum_{a(A+B)=n} 1$$

$$= 2\sum_{(4a+b)A+bB=n} 1 + \sum_{a|n}\left(\sum_{A+B=n/a} 1\right)$$

$$= 2\sum_{4aA+b(A+B)=n} 1 + \sum_{a|n}\left(\frac{n}{a}-1\right)$$

$$= 2\sum_{\substack{4aA+bB=n\\ A<B}} 1 + \sigma(n) - \tau(n)$$

$$= 2A_1 + \sigma(n) - \tau(n) \qquad ($$

and

$$\sum_{\substack{aA+bB=n\\ a\equiv -b(4)}} 1 = \sum_{\substack{aA+bB=n\\ A<B\\ a\equiv -b(4)}} 1 + \sum_{\substack{aA+bB=n\\ A>B\\ a\equiv -b(4)}} 1 + \sum_{\substack{aA+bB=n\\ A=B\\ a\equiv -b(4)}} 1$$

$$= 2\sum_{\substack{aA+bB=n\\ A<B\\ a\equiv -b(4)}} 1 + \sum_{\substack{A(a+b)=n\\ a+b\equiv 0(4)}} 1$$

$$= 2\sum_{\substack{aA+b(A+B)=n\\ a+b\equiv 0(4)}} 1 + \sum_{c|n/4}\left(\sum_{a+b=4c} 1\right)$$

$$= 2\sum_{\substack{(a+b)A+bB=n\\ a+b\equiv 0(4)}} 1 + \sum_{c|n/4}(4c-1)$$

$$= 2 \sum_{\substack{4aA+bB=n \\ 4a>b}} 1 + 4\sigma(n/4) - \tau(n/4)$$

$$= 2B_1 + 4\sigma(n/4) - \tau(n/4). \tag{13}$$

Hence, by (3), (9), (12) and (13), we have

$$\frac{1}{16}(r_4(n) - 2r_2(n)) = 2(A_1 - B_1) + (\sigma(n) - 4\sigma(n/4)) - (\tau(n) - \tau(n/4))$$

$$= 2(A_1 - B_1) + (\sigma(n) - 4\sigma(n/4)) - (\tau_1 + \tau_2 + \tau_3).$$

Recalling from (3) that $r_2(n) = 4(\tau_1 - \tau_3)$, we obtain

$$r_4(n) = 32(A_1 - B_1) + 16(\sigma(n) - 4\sigma(n/4)) - 8(\tau_1 + 2\tau_2 + 3\tau_3). \tag{14}$$

From (1) and (14) we see that it remains to show that

$$A_1 - B_1 = \frac{1}{4}(\tau_1 + 2\tau_2 + 3\tau_3) - \frac{1}{4}(\sigma(n) - 4\sigma(n/4)). \tag{15}$$

From (10) and (11), we have

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3$$

and

$$A_2 = \sum_{4a(A+B)+bB=n} 1 = \sum_{4aA+(4a+b)B=n} 1 = \sum_{\substack{4aA+bB=n \\ 4a<b}} 1 = B_2$$

so that

$$A_1 - B_1 = B_3 - A_3.$$

Finally,

$$A_3 = \sum_{(4a+b)A=n} 1 = \sum_{A|n} \left(\sum_{4a+b=n/A} 1 \right)$$

$$= \sum_{A|n} \left[\frac{n}{4A} \right]^* = \sum_{A|n} \left[\frac{A}{4} \right]^*$$

$$= \sum_{\substack{A|n \\ A\equiv 0(4)}} \left(\frac{A}{4}-1\right) + \sum_{\substack{A|n \\ A\equiv 1(4)}} \frac{A-1}{4} + \sum_{\substack{A|n \\ A\equiv 2(4)}} \frac{A-2}{4} + \sum_{\substack{A|n \\ A\equiv 3(4)}} \frac{A-3}{4}$$

$$= \frac{1}{4}\sigma(n) - \tau_0 - \frac{1}{4}\tau_1 - \frac{1}{2}\tau_2 - \frac{3}{4}\tau_3$$

and

$$B_3 = \sum_{4a(A+B)=n} 1 = \sum_{a|n/4}\left(\sum_{A+B=n/4a} 1\right)$$

$$= \sum_{a|n/4}\left(\frac{n}{4a}-1\right) = \sigma(n/4) - \tau(n/4)$$

$$= \sigma(n/4) - \tau_0.$$

This completes the proof of (15) and thus of Jacobi's four squares theorem.

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