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# The Chowla–Selberg formula for genera

by

JAMES G. HUARD (Buffalo, N.Y.), PIERRE KAPLAN (Nancy) and KENNETH S. WILLIAMS (Ottawa, Ont.)

**1. Introduction.** A nonzero integer $D$ is called a *discriminant* if $D \equiv 0$ or $1 \pmod 4$. We set

$$D = \Delta(D) f(D)^2, \tag{1.1}$$

where $f(D)$ is the largest positive integer such that $\Delta(D) = D/f(D)^2$ is a discriminant. The discriminant $D$ is called *fundamental* if $f(D) = 1$. The discriminant $\Delta(D)$ is fundamental, and is called the fundamental discriminant of the discriminant $D$. The integer $f(D)$ is called the *conductor* of the discriminant $D$. The strict equivalence classes of primitive, integral, binary quadratic forms $(a, b, c) = ax^2 + bxy + cy^2$ of discriminant $D = b^2 - 4ac$ (only positive-definite forms are used if $D < 0$) form a finite abelian group under composition. We denote this group by $H(D)$ and its order by $h(d)$. The class of the form $(a, b, c)$ is denoted by $[a, b, c]$. If $D < 0$ we set as usual

$$w(D) = \begin{cases} 6 & \text{if } D = -3, \\ 4 & \text{if } D = -4, \\ 2 & \text{if } D < -4. \end{cases} \tag{1.2}$$

The Dedekind eta function $\eta(z)$ is defined for all complex numbers $z = x + iy$ with $y > 0$ by

$$\eta(z) = e^{\pi i z/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z}). \tag{1.3}$$

We note for future reference that $\eta(iy)$ and $e^{-\pi i/24}\eta\big(\frac{1+iy}{2}\big)$ are positive numbers.

From this point on, $d$ denotes a negative discriminant, and we set $\Delta =$

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$\Delta(d)$, $f = f(d)$, so that

$$d = \Delta f^2. \tag{1.4}$$

If $[a,b,c] = [a_1,b_1,c_1] \in H(d)$, a simple calculation, using the basic properties of the Dedekind eta function (given for example in [15, §34, 38]) shows that

$$a^{-1/4}|\eta((b+\sqrt{d})/(2a))| = a_1^{-1/4}|\eta((b_1+\sqrt{d})/(2a_1))|,$$

so that the quantity $a^{-1/4}|\eta((b+\sqrt{d})/(2a))|$ depends only on the class of the form $(a,b,c)$, and thus

$$\prod_{[a,b,c]\in H(d)} a^{-1/4}|\eta((b+\sqrt{d})/(2a))|$$

is well-defined. The famous Chowla–Selberg formula [12, formula (2), p. 110] asserts that if $d$ is a fundamental discriminant then

$$\prod_{[a,b,c]\in H(d)} a^{-1/4}|\eta((b+\sqrt{d})/(2a))| = (2\pi|d|)^{-h(d)/4}\Big\{\prod_{m=1}^{|d|}(\Gamma(m/|d|))^{(\frac{d}{m})}\Big\}^{w(d)/8}, \tag{1.5}$$

where $\Gamma(z)$ is the gamma function and $\left(\frac{d}{m}\right)$ is the Kronecker symbol for discriminant $d$. This formula has been extended to arbitrary discriminants $d$ by Kaneko [8], Nakkajima and Taguchi [10] and by Kaplan and Williams [9], who showed that

$$\prod_{[a,b,c]\in H(d)} a^{-1/4}|\eta((b+\sqrt{d})/(2a))| = (2\pi|d|)^{-h(d)/4}\Big\{\prod_{m=1}^{|\Delta|}\Gamma(m/|\Delta|)^{(\frac{\Delta}{m})}\Big\}^{\frac{w(\Delta)h(d)}{8h(\Delta)}}\Big\{\prod_{p\mid f}p^{\alpha_p(\Delta,f)}\Big\}^{h(d)/4}, \tag{1.6}$$

where $p$ runs through the primes dividing $f$, $p^{v_p(f)}$ is the largest power of $p$ dividing $f$, and

$$\alpha_p(\Delta,f) = \frac{(p^{v_p(f)}-1)\big(1-\left(\frac{\Delta}{p}\right)\big)}{p^{v_p(f)-1}(p-1)\big(p-\left(\frac{\Delta}{p}\right)\big)}.$$

We remark that $p$ always denotes a prime in this paper.

The cosets of the subgroup of squares in $H(d)$ are called *genera*, and we denote the group of genera of discriminant $d$ by $G(d)$. The identity element of $G(d)$ is called the *principal genus*. It is known that the order of $G(d)$ is $2^{t(d)}$, where $t(d)$ is a nonnegative integer. When $d$ is fundamental, Williams and Zhang [16] have extended the Chowla–Selberg formula to genera. They have shown for $G \in G(d)$ ($d$ fundamental) that

$$(1.7)\qquad \prod_{[a,b,c]\in G} a^{-1/4}|\eta((b+\sqrt{d})/(2a))|$$
$$= (2\pi|d|)^{-h(d)/2^{t(d)+2}}\Big\{\prod_{m=1}^{|\Delta|}\Gamma(m/|\Delta|)^{(\frac{\Delta}{m})}\Big\}^{\frac{w(\Delta)h(d)}{2^{t(d)+3}h(\Delta)}}$$
$$\times \prod_{\substack{d_1\in F(d)\\ d_1>1}} \varepsilon_{d_1}^{\frac{-w(d_1)\gamma_{d_1}(G)h(d_1)h(d/d_1)}{w(d/d_1)2^{t(d)+1}}},$$

where $\varepsilon_{d_1}$ denotes the fundamental unit ($>1$) of the real quadratic field $\mathbb{Q}(\sqrt{d_1})$ of discriminant $d_1$, $\gamma_{d_1}(G)$ ($=\pm1$) is defined in (2.8), and the set $F(d)$ is defined in Definition 2.1. If we multiply formula (1.7) over all the $2^{t(d)}$ genera $G$ of $G(d)$, we obtain the original formula (1.5) of Chowla and Selberg as

$$\sum_{G\in G(d)} \gamma_{d_1}(G) = 0 \quad \text{for } d_1 > 1$$

(see (2.13)).

In this paper we extend the Chowla–Selberg formula for genera to arbitrary discriminants $d$. We prove

THEOREM 1.1. *For any negative discriminant $d$ and any $G \in G(d)$, we have*

$$\prod_{[a,b,c]\in G} a^{-1/4}|\eta((b+\sqrt{d})/(2a))|$$
$$= (2\pi|d|)^{-h(d)/2^{t(d)+2}}\Big\{\prod_{m=1}^{|\Delta|}\Gamma(m/|\Delta|)^{(\frac{\Delta}{m})}\Big\}^{\frac{w(\Delta)h(d)}{2^{t(d)+3}h(\Delta)}}$$
$$\times \Big\{\prod_{p\mid f} p^{\alpha_p(\Delta,f)}\Big\}^{h(d)/2^{t(d)+2}} \prod_{\substack{d_1\in F(d)\\ d_1>1}} \varepsilon_{d_1}^{\beta(d_1,d,G)},$$

*where*

$$\beta(d_1,d,G)$$
$$= \frac{-w(d)\gamma_{d_1}(G)f(d/d_1)h(d_1)h(\Delta(d/d_1))}{w(\Delta(d/d_1))2^{t(d)+1}}$$
$$\times \sum_{m\mid f(d/d_1)} \frac{1}{m} \prod_{\substack{p\mid m\\ p\nmid f/m}} \left(1-\frac{(\frac{\Delta}{p})}{p}\right) \prod_{p\mid f/m}\left(1-\frac{(\frac{d_1}{p})}{p}\right)\left(1-\frac{(\frac{\Delta(d/d_1)}{p})}{p}\right).$$

In order to prove this theorem, we first derive an explicit formula for the number $R_G(n,d)$ of representations of an arbitrary positive integer $n$ by the classes of a given genus $G$ of discriminant $d$ (see Theorem 8.1). We recall that an integer $n$ is said to be *represented by the form* $(a,b,c)$ if there exist integers $x$ and $y$ such that

$$n = ax^2 + bxy + cy^2.$$

We set

$$R_{(a,b,c)}(n,d) = \text{card}\{(x,y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n\}, \tag{1.8}$$

where we have included the discriminant $d = b^2 - 4ac$ in the notation for use later on. If the forms $(a,b,c)$ and $(a',b',c')$ belong to the same class $K \in H(d)$, then $R_{(a,b,c)}(n,d) = R_{(a',b',c')}(n,d)$. We denote this number by $R_K(n,d)$ so that, for any form $(a,b,c)$, we have

$$R_{[a,b,c]}(n,d) = R_{(a,b,c)}(n,d). \tag{1.9}$$

If $G$ is a genus in $G(d)$, we set

$$R_G(n,d) = \sum_{K\in G} R_K(n,d). \tag{1.10}$$

We also set

$$N(n,d) = \sum_{G\in G(d)} R_G(n,d) = \sum_{K\in H(d)} R_K(n,d). \tag{1.11}$$

The formula for $R_G(n,d)$ given in Theorem 8.1 shows that the Dirichlet series $\sum_{n=1}^{\infty} R_G(n,d)/n^s$ converges for $s > 1$ and can be expressed as a finite linear combination of products of pairs of Dirichlet $L$-series (Theorem 10.1). Our main result (Theorem 1.1) then follows by applying Kronecker's limit formula (see for example [13, Theorem 1, p. 14).

We conclude this introduction by indicating some instances when Theorem 1.1 can be used to evaluate some elliptic integrals of the first kind. We recall that for $0 < k < 1$ the *complete elliptic integral of the first kind* $K(k)$ is defined by

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1.12}$$

The elliptic integral $K(k)$ can be determined for certain values of $k$ as follows: let $\lambda > 0$ be such that the values of $\eta(\sqrt{-\lambda}) = A$ and $\eta(\sqrt{-\lambda}/2) = B$ are known explicitly, then

$$K(k) = \frac{\pi}{\sqrt{k}} \cdot \frac{A^4}{B^2}, \tag{1.13}$$

where $k$ is given by

$$\frac{4(1-k^2)}{k} = \frac{B^{12}}{A^{12}}, \quad 0 < k < 1 \tag{1.14}$$

(see for example [15, p. 114], [17, eqns. (2.3)–(2.8)]). Following Zucker [17] we set $K[\sqrt{\lambda}] = K(k)$. We remark that in view of the relations

$$e^{-\pi i/3}\eta^8\left(\frac{1+\sqrt{-\lambda}}{2}\right) = \eta^8\left(\frac{\sqrt{-\lambda}}{2}\right) + 16\eta^8(2\sqrt{-\lambda}) \tag{1.15}$$

and

$$\eta\left(\frac{\sqrt{-\lambda}}{2}\right)\eta\left(\frac{1+\sqrt{-\lambda}}{2}\right)\eta(2\sqrt{-\lambda}) = e^{\pi i/24}\eta^3(\sqrt{-\lambda}), \tag{1.16}$$

it is enough to know two of

$$\eta\left(\frac{\sqrt{-\lambda}}{2}\right),\ \eta\left(\frac{1+\sqrt{-\lambda}}{2}\right),\ \eta(\sqrt{-\lambda}),\ \eta(2\sqrt{-\lambda})$$

in order to be able to determine $A$ and $B$. We now give two situations when Theorem 1.1 can be used to determine $A$ and $B$.

The first occurs when $H(4d)$ has one class per genus. There are 27 known values of $d$ for which this occurs, namely, $-d = 3, 4, 7, 8, 12, 15, 16, 24, 28, 40, 48, 60, 72, 88, 112, 120, 168, 232, 240, 280, 312, 408, 520, 760, 840, 1320, 1848$ [4, pp. 88–89]. In this case $H(d)$ also has one class per genus, and applying Theorem 1.1 to the principal genus in each case, we obtain $\eta(\sqrt{d})$ and either $\eta(\sqrt{d}/2)$ or $\eta((1+\sqrt{d})/2)$ according as $d \equiv 0 \pmod 4$ or $d \equiv 1 \pmod 4$. Thus we can determine $K[\sqrt{-d}]$. Two simple numerical examples are provided by $d = -4$ ($\lambda = 4$) and $d = -3$ ($\lambda = 3$). For $d = -4$, from Theorem 1.1, we deduce

$$A = \eta(\sqrt{-4}) = 2^{-9/8}\pi^{-1/4}\left\{\frac{\Gamma(1/4)}{\Gamma(3/4)}\right\}^{1/2},$$

$$B = \eta(\sqrt{-1}) = 2^{-3/4}\pi^{-1/4}\left\{\frac{\Gamma(1/4)}{\Gamma(3/4)}\right\}^{1/2},$$

and then from (1.14) and (1.13) we obtain $k = 3 - 2\sqrt{2}$ and

$$K[2] = K(3-2\sqrt{2}) = \frac{(\sqrt{2}+1)\pi^{1/2}}{2^3}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)} = \frac{(\sqrt{2}+1)}{2^{7/2}\pi^{1/2}}\Gamma^2(1/4).$$

When $d = -3$ by Theorem 1.1 we have

$$A = \eta(\sqrt{-3}) = 2^{-7/12}3^{-1/4}\pi^{-1/4}\left\{\frac{\Gamma(1/3)}{\Gamma(2/3)}\right\}^{3/4},$$

$$\eta\left(\frac{1+\sqrt{-3}}{2}\right) = e^{\pi i/24}2^{-1/4}3^{-1/4}\pi^{-1/4}\left\{\frac{\Gamma(1/3)}{\Gamma(2/3)}\right\}^{3/4}.$$

From (1.15) and (1.16) we obtain

$$B=\eta\left(\frac{\sqrt{-3}}{2}\right)=2^{-5/8}3^{-1/4}\pi^{-1/4}(1+\sqrt{3})^{1/4}\left\{\frac{\Gamma(1/3)}{\Gamma(2/3)}\right\}^{3/4},$$

$$\eta(2\sqrt{-3})=2^{-7/8}3^{-1/4}\pi^{-1/4}(1+\sqrt{3})^{-1/4}\left\{\frac{\Gamma(1/3)}{\Gamma(2/3)}\right\}^{3/4}.$$

Then, from (1.14), we deduce that $k=(\sqrt{6}-\sqrt{2})/4$, and, from (1.13), we obtain

$$K[\sqrt{3}]=\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)=2^{-5/6}3^{-1/2}\pi^{1/2}\left\{\frac{\Gamma(1/3)}{\Gamma(2/3)}\right\}^{3/2}$$
$$=2^{-7/3}3^{1/4}\pi^{-1}\{\Gamma(1/3)\}^{3}.$$

These values of $K$ are in agreement with [1, Table 9.1, p. 298 and p. 139], where the values of $K[\sqrt{\lambda}]$ are given for $\lambda=1,2,\ldots,16$. Similarly we can determine $K[\sqrt{7}],K[\sqrt{8}],K[\sqrt{12}],\ldots,K[\sqrt{1848}]$.

The second situation occurs when $H(d)$ ($d\equiv 8 \pmod{16}$) has one class per genus with the classes $[1,0,-d/4]$ and $[2,0,-d/8]$ in different genera. It is known that this occurs for $d=-24,-40,-72,-88,-120,-168,-232,-280,-312,-408,-520,-760,-840,-1320,-1848$ (see [4]). Applying Theorem 1.1 to these genera, we obtain, for $\lambda=-d/4$,

$$A=\eta(\sqrt{d}/2)=\eta(\sqrt{-\lambda}),\qquad B=\eta(\sqrt{d}/4)=\eta(\sqrt{-\lambda}/2).$$

We illustrate this situation with an example not given in Table 9.1 of [1]. We take $d=-88$, so that $\lambda=22$. Here $H(-88)=\{[1,0,22],[2,0,11]\}$ and the class $[2,0,11]$ is not in the principal genus. Applying Theorem 1.1 to the classes $[1,0,22]$ and $[2,0,11]$, we obtain

$$A=\eta(\sqrt{-22})=2^{-1}11^{-1/4}\pi^{-1/4}E^{1/8}(1+\sqrt{2})^{-1/4}$$

and

$$B=\eta\left(\frac{\sqrt{-22}}{2}\right)=2^{-3/4}11^{-1/4}\pi^{-1/4}E^{1/8}(1+\sqrt{2})^{1/4},$$

where

$$E=\prod_{m=1}^{88}\Gamma\left(\frac{m}{88}\right)^{\left(\frac{-88}{m}\right)}.$$

Then, from (1.14), we obtain $k=(1+\sqrt{2})^3(3\sqrt{22}-7-5\sqrt{2})$, so that

$$\frac{1}{\sqrt{k}}=(1+\sqrt{2})^{3/2}(7+5\sqrt{2}+3\sqrt{22})^{1/2},$$

and thus, by (1.13),

$$K[\sqrt{22}] = K(-99 - 70\sqrt{2} + 30\sqrt{11} + 21\sqrt{22})$$
$$= 2^{-5/2} 11^{-1/2} (7 + 5\sqrt{2} + 3\sqrt{22})^{1/2} \pi^{1/2} \left\{ \prod_{m=1}^{88} \Gamma\left(\frac{m}{88}\right)^{(\frac{-88}{m})} \right\}^{1/4}.$$

In a similar manner we can determine $K[\sqrt{6}], K[\sqrt{10}], K[\sqrt{18}], K[\sqrt{30}], \ldots$ $\ldots, K[\sqrt{462}]$.

**2. Prime discriminants and genera.** An odd prime discriminant is a discriminant of the form $p^* = (-1)^{(p-1)/2} p$, where $p$ is an odd prime. The discriminants $-4, 8, -8$ are called *even prime discriminants*. We now define the prime discriminants corresponding to the discriminant $d$, and note some of their properties.

Definition 2.1. (a) The *prime discriminants corresponding to the discriminant* $d$ are the discriminants $p_1^*, \ldots, p_{t+1}^*$, together with $p_{t+2}^*$ if $d \equiv 0 \pmod{32}$, where $t = t(d)$ and $|G(d)| = 2^t$, given as follows:

(i) $d \equiv 1 \pmod 4$ or $d \equiv 4 \pmod{16}$
$p_1 < p_2 < \ldots < p_{t+1}$ are the odd prime divisors of $d$.

(ii) $d \equiv 12 \pmod{16}$ or $d \equiv 16 \pmod{32}$
$p_1 < p_2 < \ldots < p_t$ are the odd prime divisors of $d$ and $p_{t+1}^* = -4$.

(iii) $d \equiv 8 \pmod{32}$
$p_1 < p_2 < \ldots < p_t$ are the odd prime divisors of $d$ and $p_{t+1}^* = 8$.

(iv) $d \equiv 24 \pmod{32}$
$p_1 < p_2 < \ldots < p_t$ are the odd prime divisors of $d$ and $p_{t+1}^* = -8$.

(v) $d \equiv 0 \pmod{32}$
$p_1 < p_2 < \ldots < p_{t-1}$ are the odd prime divisors of $d$, $p_t^* = -4$, $p_{t+1}^* = 8$, and $p_{t+2}^* = -8$.

(b) The set of prime discriminants corresponding to $d$ is denoted by $P(d)$. We note that these are coprime in pairs if $d \not\equiv 0 \pmod{32}$. The set of all products of pairwise coprime elements of $P(d)$ is denoted by $F(d)$.

It is known that a fundamental discriminant $d$ can be expressed uniquely as a product of prime discriminants, and moreover these prime discriminants are precisely the elements of $P(d)$.

Lemma 2.1. (a) $F(d) = \{d_1 : d_1$ *is a fundamental discriminant*, $d_1 \mid d$, *and* $d/d_1$ *is a discriminant*$\}$.

(b) *For any positive integer* $k$, $P(d) \subseteq P(dk^2)$ *and* $F(d) \subseteq F(dk^2)$. *Also*,

$$P(\Delta) \subseteq P(d), \quad 1 \in F(d), \quad \Delta \in F(d), \quad |F(d)| = 2^{t(d)+1},$$
$$|P(d)| = \begin{cases} t(d)+1 & \text{if } d \not\equiv 0 \pmod{32}, \\ t(d)+2 & \text{if } d \equiv 0 \pmod{32}. \end{cases}$$

Proof. The assertions of Lemma 2.1 are straightforward consequences of Definition 2.1. ■

We now recall the definition of the Legendre–Jacobi–Kronecker symbol $\left(\frac{D}{k}\right)$ for a discriminant $D$ and a positive integer $k$ (see for example [3, pp. 18–21, 35]). For $p$ an odd prime

$$\left(\frac{D}{p}\right) = \begin{cases} +1 & \text{if } D \text{ is a nonzero square} \pmod p, \\ -1 & \text{if } D \text{ is not a square} \pmod p, \\ 0 & \text{if } p \mid D; \end{cases} \tag{2.1}$$

$$\left(\frac{D}{2}\right) = \begin{cases} +1 & \text{if } D \equiv 1 \pmod 8, \\ -1 & \text{if } D \equiv 5 \pmod 8, \\ 0 & \text{if } D \equiv 0 \pmod 4; \end{cases} \tag{2.2}$$

and generally

$$\left(\frac{D}{k}\right) = \prod_{p\mid k} \left(\frac{D}{p}\right)^{v_p(k)}. \tag{2.3}$$

Next we recall some of the properties of genera. The basic properties of generic characters and genera can be found for example in [2], [6]. Let $p^* \in P(d)$ and $K \in H(d)$. For any positive integer $k$ coprime with $p^*$ represented by $K$, it is known that $\left(\frac{p^*}{k}\right)$ has the same value, so we can set

$$\gamma_{p^*}(K) = \left(\frac{p^*}{k}\right) = \pm 1.$$

Let $G \in G(d)$. Genus theory shows that, for any $K \in G$, $\gamma_{p^*}(K)$ has the same value, so we can set $\gamma_{p^*}(G) = \gamma_{p^*}(K)$, and furthermore that

$$\gamma_{p^*}(G_1G_2) = \gamma_{p^*}(G_1)\gamma_{p^*}(G_2), \tag{2.4}$$

for $G_1, G_2 \in G(d)$. One of the main results of genus theory is the product formula (2.5) (see for example [6, equation (9)]).

Lemma 2.2. *If $G \in G(d)$ then, with $\Delta = \Delta(d)$,*

$$\prod_{p^* \in P(\Delta)} \gamma_{p^*}(G) = 1, \tag{2.5}$$

*together with*

$$\gamma_{-4}(G)\gamma_8(G)\gamma_{-8}(G) = 1 \quad \text{if } d \equiv 0 \pmod{32}. \tag{2.6}$$

*Moreover, if $\delta_{p^*} = \pm 1$ for each $p^* \in P(d)$ and*

$$\prod_{p^* \in P(\Delta)} \delta_{p^*} = 1, \tag{2.7}$$

*together with*

$$\delta_{-4}\delta_8\delta_{-8} = 1 \quad \text{if } d \equiv 0 \pmod{32}, \tag{2.8}$$

*then there exists a unique* $G \in G(d)$ *with*

$$\gamma_{p^*}(G) = \delta_{p^*} \quad \textit{for each } p^* \in P(d). \tag{2.9}$$

We observe that Lemma 2.2 is consistent with

$$|G(d)| = \begin{cases} \dfrac{1}{2} \cdot 2^{|P(d)|} = \dfrac{1}{2} \cdot 2^{t(d)+1} = 2^{t(d)} & \text{if } d \not\equiv 0 \pmod{32}, \\ \dfrac{1}{2^2} \cdot 2^{|P(d)|} = \dfrac{1}{2^2} \cdot 2^{t(d)+2} = 2^{t(d)} & \text{if } d \equiv 0 \pmod{32}, \end{cases}$$

and shows also that there are exactly $2^{|P(d)|-|P(\Delta)|} = 2^{t(d)-t(\Delta)}$ genera $G$ in $G(d)$ with $\gamma_{p^*}(G) = \delta_{p^*}$ for each $p^* \in P(\Delta)$.

We now extend the definition of $\gamma_{p^*}(G)$ ($p^* \in P(d)$) to $\gamma_{d_1}(G)$ for $d_1 \in F(d)$. For $d_1 \in F(d)$, we set

$$\gamma_{d_1}(G) = \prod_{p^* \in P(d_1)} \gamma_{p^*}(G) = \pm 1. \tag{2.10}$$

By (2.4) and (2.10) each $\gamma_{d_1}$ ($d_1 \in F(d)$) is a group character of $G(d)$, and it is known from genus theory [2, §4.3] that these include all the group characters of $G(d)$.

The set $F(d)$ is a group under the binary operation $\circ$ defined by

$$d_1 \circ d_2 = \Delta(d_1 d_2), \quad d_1, d_2 \in F(d).$$

The identity element is 1 and each element is its own inverse. As $\Delta \in F(d)$, and $d_1 \circ \Delta = \Delta(d_1 \Delta) = \Delta(d_1 d) = \Delta(d/d_1)$, the mapping

$$d_1 \to \Delta(d/d_1) \tag{2.11}$$

is a translation and thus a bijection on $F(d)$.

Let $\widehat{G(d)}$ be the group of characters of $G(d)$. The mapping $\phi : F(d) \to \widehat{G(d)}$ given by $\phi(d_1) = \gamma_{d_1}$ is easily checked to be a homomorphism using (2.6) if $d \equiv 0 \pmod{32}$. It is known from genus theory [2] that $\phi$ is surjective, and thus $|\ker \phi| = |F(d)|/|\widehat{G(d)}| = |F(d)|/|G(d)| = 2^{t(d)+1}/2^{t(d)} = 2$. By (2.5) we have $\gamma_\Delta(G) = 1$, for all $G \in G(d)$, so that $\ker \phi = \{1, \Delta\}$. Further, for $d_1 \in F(d)$, we have

$$\gamma_{\Delta(d/d_1)} = \gamma_{d_1 \circ \Delta} = \gamma_{d_1} \gamma_\Delta = \gamma_{d_1}. \tag{2.12}$$

By the theory of group characters, we have

$$\sum_{G \in G(d)} \gamma_{d_1}(G) = \begin{cases} |G(d)| = 2^{t(d)} & \text{if } d_1 = 1 \text{ or } \Delta, \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

and

$$\sum_{d_1 \in F(d)} \gamma_{d_1}(G) = \begin{cases} 2|\widehat{G(d)}| = 2^{t(d)+1} & \text{if } G \text{ is the principal genus,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

**3. The derived genus $G_m$ of $G$.** In this section we define the derived genus $G_m \in G(d/(m,f)^2)$ of $G \in G(d)$, where $m$ is a positive integer all of whose prime factors $p$ divide $d$ and satisfy

$$p \nmid \Delta \Rightarrow v_p(m) \le v_p(f). \tag{3.1}$$

We begin with the case when $m$ is a prime.

PROPOSITION 3.1. *Let $p$ be a prime with $p \mid d$, and let $G \in G(d)$. Then there is a unique genus*

$$G_p \in \begin{cases} G(d/p^2) & \text{if } p \mid f, \\ G(d) & \text{if } p \nmid f, \end{cases}$$

*such that in the case $p \mid f$,*

$$\gamma_{q^*}(G_p) = \gamma_{q^*}(G) \quad \text{for every } q^* \in P(d/p^2), \tag{3.2}$$

*and in the case $p \nmid f$ (so that $p \mid \Delta$),*

$$\gamma_{q^*}(G_p) = \begin{cases} \left(\dfrac{q^*}{p}\right)\gamma_{q^*}(G) & \text{for every } q^* \in P(d) \text{ with } p \nmid q^*, \\ \left(\dfrac{d/q^*}{p}\right)\gamma_{q^*}(G) = \left(\dfrac{\Delta/q^*}{p}\right)\gamma_{q^*}(G) & \\ & \text{for the unique } q^* \in P(d) \text{ with } p \mid q^*. \end{cases} \tag{3.3}$$

Proof. In the case $p \mid f$, we see that $d/p^2$ is a discriminant, and $P(d/p^2) \subseteq P(d)$. Hence $\gamma_{q^*}(G)$ is defined for every $q^* \in P(d/p^2)$. As $\Delta(d/p^2) = \Delta$, by Lemma 2.2, we have

$$\prod_{q^* \in P(\Delta)} \gamma_{q^*}(G) = 1,$$

together with $\gamma_{-4}(G)\gamma_8(G)\gamma_{-8}(G) = 1$, if $d \equiv 0 \pmod{32}$. Hence, by Lemma 2.2, there exists a unique genus $G_p \in G(d/p^2)$ satisfying (3.2).

We now turn to the case $p \nmid f$, so that $p \mid \Delta$. We show first that there is a unique $q^* \in P(d)$ with $p \mid q^*$. If $p \ne 2$ then $q^* = p^*$. If $p = 2$ then $2 \nmid f$ so that $d \not\equiv 0 \pmod{32}$, and thus as $2 \mid \Delta$ there is a unique $q^*$ with $2 \mid q^*$. In both cases we have $q^* \mid \Delta$. Further, as $\Delta$ is fundamental we see that $p \nmid \Delta/q^*$, so $\left(\frac{\Delta/q^*}{p}\right) = \pm 1$. Thus

$$\delta_{q^*} = \left(\frac{q^*}{p}\right)\gamma_{q^*}(G) = \pm 1 \quad \text{for every } q^* \in P(d) \text{ with } p \nmid q^*$$

and

$$\delta_{q^*} = \left(\frac{\Delta/q^*}{p}\right)\gamma_{q^*}(G) = \pm 1 \quad \text{for those } q^* \in P(d) \text{ with } p \mid q^*,$$

and we show that these $\delta_{q^*}$ satisfy the product formula (2.5). As $p \mid \Delta$ and $\Delta$ is fundamental, $\Delta$ possesses a unique prime discriminant $r^*$ with $p \mid r^*$, and

$$\prod_{q^* \in P(\Delta)} \delta_{q^*} = \left(\frac{\Delta/r^*}{p}\right)\gamma_{r^*}(G) \prod_{\substack{q^* \in P(\Delta)\\ q^* \neq r^*}} \left(\frac{q^*}{p}\right)\gamma_{q^*}(G)$$
$$= \left(\frac{\Delta/r^*}{p}\right)\left(\frac{\Delta/r^*}{p}\right) \prod_{q^* \in P(\Delta)} \gamma_{q^*}(G) = 1.$$

Further, if $d \equiv 0 \pmod{32}$, then $p \neq 2$ and

$$\delta_{-4}\delta_8\delta_{-8} = \left(\frac{-4}{p}\right)\gamma_{-4}(G)\left(\frac{8}{p}\right)\gamma_8(G)\left(\frac{-8}{p}\right)\gamma_{-8}(G)$$
$$= \left(\frac{256}{p}\right)\gamma_{-4}(G)\gamma_8(G)\gamma_{-8}(G) = 1.$$

This completes the proof of the existence of $G_p$ in this case.

Finally, we observe that for $q^* \in P(d)$ with $p \mid q^*$, we have

$$\left(\frac{\Delta/q^*}{p}\right) = \left(\frac{\Delta f^2/q^*}{p}\right) = \left(\frac{d/q^*}{p}\right). \ \blacksquare$$

Next we define $G_{p^i}$ for $p \mid d$ and $i \geq 0$. We set $G_1 = G$. By (3.2) we define successively

$$G_{p^i} = (G_{p^{i-1}})_p \in G(d/p^{2i}), \quad i = 1, \ldots, v_p(f).$$

If in addition $p \mid \Delta$, as $p \nmid f/p^{v_p(f)}$, we define successively, by (3.3),

$$G_{p^i} = (G_{p^{i-1}})_p \in G(d/p^{2v_p(f)}), \quad i = v_p(f)+1, \ldots$$

Thus, for any $p \mid d$, we have defined $G_{p^i} \in G(d/(p^i, f)^2)$ for any nonnegative integer $i$ if $p \mid \Delta$ and for $i = 0, 1, \ldots, v_p(f)$ if $p \nmid \Delta$.

It is easy to check that if $p$ and $q$ are distinct primes dividing $d$, we have $(G_p)_q = (G_q)_p \in G(d/(pq, f)^2)$, and this allows us to define the derived genus $G_m$ as follows: for $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ satisfying (3.1) set

$$G_m = (\ldots((G_{p_1^{\alpha_1}})_{p_2^{\alpha_2}})\ldots)_{p_r^{\alpha_r}} \in G(d/(m, f)^2).$$

Lemma 3.1. (a) *Let $p$ be a prime with $p \mid d$. Let $d_1 \in F(d/(p, f)^2)$. Then, for any $G \in G(d)$, we have*

$$\gamma_{d_1}(G_p) = \begin{cases} \gamma_{d_1}(G) & \text{if } p \mid f, \\ \left(\frac{d_1}{p}\right)\gamma_{d_1}(G) & \text{if } p \nmid f,\ p \nmid d_1, \\ \left(\frac{d/d_1}{p}\right)\gamma_{d_1}(G) & \text{if } p \nmid f,\ p \mid d_1. \end{cases}$$

(b) *Further, if $m$ is a positive integer with $m \mid f$, $G \in G(d)$, and $d_1 \in F(d/m^2)$, then*

$$\gamma_{d_1}(G_m) = \gamma_{d_1}(G).$$

Proof. (a) In this proof we let $P(d_1) = \{p_1^*, \ldots, p_r^*\}$ so that $d_1 = p_1^* \ldots p_r^*$ and the $p_i^*$ are coprime in pairs. We first consider the case $p \mid f$, so that $d_1 \in F(d/p^2)$, and thus each $p_i^* \in P(d/p^2)$. Then

$$\begin{aligned} \gamma_{d_1}(G_p) &= \gamma_{p_1^*}(G_p) \ldots \gamma_{p_r^*}(G_p) && \text{(by (2.10))} \\ &= \gamma_{p_1^*}(G) \ldots \gamma_{p_r^*}(G) && \text{(by (3.2))} \\ &= \gamma_{d_1}(G). && \text{(by (2.10))} \end{aligned}$$

We now turn to the case $p \nmid f$, so that $p \mid \Delta$, $d_1 \in F(d)$, and thus each $p_i^* \in P(d)$. As the $p_i^*$ are coprime in pairs, at most one of the $p_i^*$ is divisible by $p$. If $p$ does not divide any of the $p_i^*$ then

$$\begin{aligned} \gamma_{d_1}(G_p) &= \gamma_{p_1^*}(G_p) \ldots \gamma_{p_r^*}(G_p) && \text{(by (2.10))} \\ &= \left(\frac{p_1^*}{p}\right)\gamma_{p_1^*}(G) \ldots \left(\frac{p_r^*}{p}\right)\gamma_{p_r^*}(G) && \text{(by (3.3))} \\ &= \left(\frac{d_1}{p}\right)\gamma_{d_1}(G). && \text{(by (2.10))} \end{aligned}$$

If $p$ divides one of the $p_i^*$, say $p_r^*$, then

$$\begin{aligned} \gamma_{d_1}(G_p) &= \gamma_{p_1^*}(G_p) \ldots \gamma_{p_r^*}(G_p) && \text{(by (2.10))} \\ &= \left(\frac{p_1^*}{p}\right)\gamma_{p_1^*}(G) \ldots \left(\frac{p_{r-1}^*}{p}\right)\gamma_{p_{r-1}^*}(G)\left(\frac{d/p_r^*}{p}\right)\gamma_{p_r^*}(G) && \text{(by (3.3))} \\ &= \left(\frac{d_1/p_r^*}{p}\right)\left(\frac{d/p_r^*}{p}\right)\gamma_{d_1}(G) && \text{(by (2.10))} \\ &= \left(\frac{d/d_1}{p}\right)\gamma_{d_1}(G), \end{aligned}$$

as

$$\left(\frac{d_1/p_r^*}{p}\right)\left(\frac{d/p_r^*}{p}\right) = \left(\frac{d_1/p_r^*}{p}\right)^2\left(\frac{d/d_1}{p}\right) = \left(\frac{d/d_1}{p}\right).$$

(b) As $m \mid f$ the asserted result follows by applying part (a) to each prime dividing $m$ taking into account multiplicity. ∎

**4. Null primes and the integers $M, Q$ and $U$.** It is convenient to introduce the following positive integers:

(4.1) $M = M(n,d)$ is the largest integer such that $M^2 \mid n$, $M \mid f$,

(4.2) $$U = U(n,d) = \prod_{p \mid d,\, p \nmid f} p^{v_p(n)},$$

$$Q = Q(n,d) = U(n/M^2, d/M^2) = \prod_{p \mid d/M^2,\, p \nmid f/M} p^{v_p(n/M^2)}. \tag{4.3}$$

Definition 4.1. A prime $p$ is said to be a *null prime with respect to $n$ and $d$* if

$$v_p(n) \equiv 1 \pmod 2, \qquad v_p(n) < 2v_p(f). \tag{4.4}$$

The set of all such null primes is denoted by $\mathrm{Null}(n,d)$.

Proposition 4.1. *If* $\mathrm{Null}(n,d) \neq \emptyset$ *then* $N(n,d) = 0$, *where* $N(n,d)$ *is defined in* (1.11).

Proof. We suppose that $\mathrm{Null}(n,d) \neq \emptyset$ and that $N(n,d) > 0$. Let $p \in \mathrm{Null}(n,d)$. As $N(n,d) > 0$, there exists a form $(a,b,c)$ with $b^2 - 4ac = d$, where we may suppose that $(a,p) = 1$, and integers $x, y$ such that

$$n = ax^2 + bxy + cy^2.$$

Completing the square, we obtain

$$4an = X^2 - \Delta f^2 y^2, \qquad \text{where } X = 2ax + by.$$

Set $m = v_p(n)$, so that, by (4.4), $m$ is odd and $p^{m+1} \mid f^2$. As $p \nmid a$ we see that $v_p(4an)$ is odd, and thus $y \neq 0$. We now consider two cases according as $v_p(\Delta f^2 y^2)$ is odd or even.

In the former case we must have $v_p(4an) = v_p(\Delta f^2 y^2)$. If $p \neq 2$ then $v_p(4an) = m$ and $v_p(\Delta f^2 y^2) \geq m+1$, a contradiction. If $p = 2$, then $v_2(4an) = 2+m$ and, as $v_2(\Delta)$ is odd and so equal to 3, we have $v_2(\Delta f^2 y^2) \geq 3 + (m+1)$, a contradiction.

In the latter case we see that $X \neq 0$ and $v_p(X^2) = v_p(\Delta f^2 y^2)$. If $p \neq 2$ then $v_p(X^2) = v_p(\Delta f^2 y^2) \geq m+1$ so that $v_p(4an) \geq m+1$, contradicting $v_p(4an) = m$. If $p = 2$ then $v_2(\Delta)$ is even, and thus $v_2(\Delta) = 0$ or $2$. If $v_2(\Delta) = 2$ then $v_2(X^2) = v_2(\Delta f^2 y^2) \geq 2 + (m+1)$, so $v_2(4an) \geq m+3$; if $v_2(\Delta) = 0$ then $\Delta \equiv 1 \pmod 4$, and setting $v_2(X^2) = v_2(\Delta f^2 y^2) = 2w$, we see that $v_2((X/2^w)^2 - \Delta(fy/2^w)^2) \geq 2$, and hence $v_2(X^2 - \Delta f^2 y^2) \geq 2 + 2w \geq 2 + (m+1)$. Each instance contradicts $v_2(4an) = 2 + m$. ∎

By Proposition 4.1 and (1.11) we have $R_G(n,d) = 0$ if $\mathrm{Null}(n,d) \neq \emptyset$. Thus it remains to evaluate $R_G(n,d)$ when $\mathrm{Null}(n,d) = \emptyset$. This is done by means of two reduction formulae (Theorems 6.1 and 7.1). The next lemma gives some properties of $M$ and $Q$ when $\mathrm{Null}(n,d) = \emptyset$.

Lemma 4.1. (a) *If* $\mathrm{Null}(n,d) = \emptyset$ *then*

$$(n/M^2, f/M) = 1 \tag{4.5}$$

*and*

$$(n/M^2Q, d/M^2) = 1. \tag{4.6}$$

(b) $(n, f) = 1 \Leftrightarrow \mathrm{Null}(n, d) = \emptyset$ *and* $M = 1$.

P r o o f. (a) Suppose $\mathrm{Null}(n, d) = \emptyset$ but $(n/M^2, f/M) > 1$. Then there exists a prime $p$ with $p \mid n/M^2$ and $p \mid f/M$. By the maximality of $M$, we have $p^2 \nmid n/M^2$ so that $p \parallel n/M^2$. Thus $v_p(n) = 1 + 2v_p(M) < 2 + 2v_p(M) \le 2v_p(f)$, showing that $p \in \mathrm{Null}(n, d)$, a contradiction. This proves (4.5).

Suppose now there exists a prime $q$ with $q \mid n/M^2Q$ and $q \mid d/M^2$. Then, as $(n/M^2, f/M) = 1$, we have $q \nmid f/M$, so $v_q(Q) = v_q(n/M^2)$, contradicting $q \mid n/M^2Q$. This proves (4.6).

(b) Suppose $(n, f) = 1$. By definition we have $M = 1$. Now suppose that $p \in \mathrm{Null}(n, d)$. Then $v_p(n)$ is odd and $v_p(n) < 2v_p(f)$. Thus $p \mid n$ and so $p \nmid f$, a contradiction.

Now suppose that $\mathrm{Null}(n, d) = \emptyset$ and $M = 1$. By (4.5) we have $(n, f) = 1$. ∎

**5. The sum $S(n, d_1, d/d_1)$.** In this section we introduce the sum $S(n, d_1, d/d_1)$ in terms of which we give our formula for $R_G(n, d)$ (Theorem 8.1). Before giving the definition we recall from Lemma 2.1(a) that for $d_1 \in F(d)$ both $d_1$ and $d/d_1$ are discriminants.

For $d_1 \in F(d)$ and $(n, f) = 1$, we set

$$S(n, d_1, d/d_1) = \sum_{\mu\nu = n} \left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right), \tag{5.1}$$

where $\mu$ and $\nu$ run through all positive integers with $\mu\nu = n$.

LEMMA 5.1. *Suppose $(n, f) = 1$. Let $p$ be a prime such that $p \mid n$ and $p \mid d$. Then, for $G \in G(d)$, we have*

$$\sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1) = \sum_{d_1 \in F(d)} \gamma_{d_1}(G_p) S(n/p, d_1, d/d_1).$$

P r o o f. Clearly $(n/p, f) = 1$ so that $S(n/p, d_1, d/d_1)$ is defined. We have

$$\sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1)$$
$$= \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \sum_{\mu\nu = n} \left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right)$$

$$= \sum_{d_1\in F(d)} \gamma_{d_1}(G)\Big\{\sum_{\substack{\mu\nu=n\\ p\mid\mu}} \left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right)$$
$$+ \sum_{\substack{\mu\nu=n\\ p\mid\nu}} \left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right) - \sum_{\substack{\mu\nu=n\\ p\mid\mu,\, p\mid\nu}} \left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right)\Big\}$$
$$= \sum_{d_1\in F(d)} \gamma_{d_1}(G)\left(\frac{d_1}{p}\right) S(n/p, d_1, d/d_1)$$
$$+ \sum_{d_1\in F(d)} \gamma_{d_1}(G)\left(\frac{d/d_1}{p}\right) S(n/p, d_1, d/d_1).$$

In the first sum we need only sum over those $d_1$ satisfying $p\nmid d_1$, and in the second sum over those $d_1$ satisfying $p\nmid d/d_1$, equivalently, $p\mid d_1$. The result now follows by appealing to Lemma 3.1(a) as $p\nmid f$. ■

LEMMA 5.2. *Suppose* $(n,f)=1$. *Then*, *for* $G\in G(d)$, *we have*

$$\sum_{d_1\in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1) = \sum_{d_1\in F(d)} \gamma_{d_1}(G_U) S(n/U, d_1, d/d_1),$$

*where* $U$ *is defined in* (4.2).

Proof. This follows immediately from Lemma 5.1 by applying it to all primes $p$ dividing $U$ with multiplicity taken into account. ■

**6. First reduction formula.** Our first reduction formula relates $R_G(n,d)$ to $R_{G_M}(n/M^2, d/M^2)$, where $M$ is defined in (4.1).

THEOREM 6.1. *For* $G\in G(d)$, *we have*

$$R_G(n,d) = \frac{1}{2^{t(d)-t(d/M^2)}}\cdot\frac{h(d)}{h(d/M^2)} R_{G_M}(n/M^2, d/M^2).$$

In order to prove this result we need a number of lemmas.

LEMMA 6.1. *Suppose that* $p\mid f$. *Let* $K\in H(d)$. *Then*

(a) *$K$ contains a form* $(a,b,c)$ *with* $p\nmid a$, $p\mid b$ *and* $p^2\mid c$;

(b) *the mapping* $\theta_p : H(d)\to H(d/p^2)$ *given by* $\theta_p([a,b,c]) = [a, b/p, c/p^2]$ *is a surjective homomorphism*;

(c) *if* $G\in G(d)$ *and* $K\in G$ *then* $\theta_p(K)\in G_p$;

(d) *the mapping* $\widetilde{\theta}_p : G(d)\to G(d/p^2)$ *given by* $\widetilde{\theta}_p(G) = G_p$ *is a surjective homomorphism.*

Proof. (a), (b). See [5, §§150–151].

(c) Let $q^* \in P(d/p^2)$, $G \in G(d)$, and $K \in G$. We can choose $a, b, c$ with $K = [a, b, c]$, $(a, pq^*) = 1$, $p \mid b$ and $p^2 \mid c$. By (b), $\theta_p(K) = [a, b/p, c/p^2]$. Clearly $a$ is represented by the class $\theta_p(K)$ and

$$\left(\frac{q^*}{a}\right) = \gamma_{q^*}(G) = \gamma_{q^*}(G_p),$$

for all $q^* \in P(d/p^2)$, so that $\theta_p(K) \in G_p$.

(d) As $\theta_p : H(d) \to H(d/p^2)$ is a surjective homomorphism and $G(d) = H(d)/H^2(d)$, $G(d/p^2) = H(d/p^2)/H^2(d/p^2)$, it follows that $\widetilde{\theta}_p : G(d) \to G(d/p^2)$ is also a surjective homomorphism. ∎

Lemma 6.2. *Let $p$ be a prime with $p \mid M$. Then, for any class $K \in H(d)$, we have*

$$R_K(n, d) = R_{\theta_p(K)}(n/p^2, d/p^2).$$

Proof. By Lemma 6.1(a) we choose $(a, b, c) \in K$ with $p \nmid a$, $p \mid b$ and $p^2 \mid c$ so that $\theta_p(K) = [a, b/p, c/p^2]$. Set

$$S = \{(x, y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n\},$$
$$T = \left\{(X, Y) \in \mathbb{Z}^2 : aX^2 + \frac{b}{p}XY + \frac{c}{p^2}Y^2 = \frac{n}{p^2}\right\},$$

and define the one-to-one mapping $\lambda : T \to S$ by $\lambda((X, Y)) = (pX, Y)$. If $(x, y) \in S$, then as $p \mid n$, we see that $p \mid x$ and $\lambda((x/p, y)) = (x, y)$. Hence $\lambda$ is onto, and thus

$$R_{(a,b,c)}(n, d) = |S| = |T| = R_{(a,b/p,c/p^2)}(n/p^2, d/p^2),$$

completing the proof. ∎

Lemma 6.3. *Let $p$ be a prime with $p \mid M$. Then, for $G \in G(d)$, we have*

$$R_G(n, d) = \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}}\, R_{G_p}(n/p^2, d/p^2).$$

Proof. Let $G \in G(d)$. There are $|\ker \widetilde{\theta}_p|$ distinct genera of $G(d)$ that are mapped to $G_p$ by $\widetilde{\theta}_p$. As $K$ runs through the classes of these genera, $\theta_p(K)$ runs through the classes of $G_p$ exactly $|\ker \theta_p|$ times. Hence, as $K$ runs through the classes of $G$, $\theta_p(K)$ runs through the classes of $G_p$ exactly $|\ker \theta_p|/|\ker \widetilde{\theta}_p|$ times. Hence

$$\begin{aligned} R_G(n, d) &= \sum_{K \in G} R_K(n, d) && \text{(by (1.10))} \\ &= \sum_{K \in G} R_{\theta_p(K)}(n/p^2, d/p^2) && \text{(by Lemma 6.2)} \end{aligned}$$

$$= \frac{|\ker \theta_p|}{|\ker \widetilde{\theta}_p|} \sum_{K' \in G_p} R_{K'}(n/p^2, d/p^2)$$

$$= \frac{h(d)/h(d/p^2)}{|G(d)|/|G(d/p^2)|} R_{G_p}(n/p^2, d/p^2) \qquad \text{(by Lemma 6.1)}$$

$$= \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}} R_{G_p}(n/p^2, d/p^2). \ \blacksquare$$

Proof of Theorem 6.1. Theorem 6.1 follows from Lemma 6.3 by applying it to all primes dividing $M$ taking multiplicity into account. ∎

**7. Second reduction formula.** Our second reduction formula removes from $n$ those primes which divide $d$ but do not divide $f$.

THEOREM 7.1. *For $G \in G(d)$, we have*

$$R_G(n, d) = R_{G_U}(n/U, d),$$

*where $U = U(n, d)$ is defined in* (4.2).

Before giving the proof of Theorem 7.1, we state and prove a number of lemmas.

LEMMA 7.1. *Suppose that $p$ is a prime with $p \mid d$ and $p \nmid f$. Let $K \in H(d)$. Then*

(a) *$K$ contains a form $(a, b, cp)$ with $p \nmid ac$ and $p \mid b$*;

(b) *the mapping $\phi_p : H(d) \to H(d)$ given by $\phi_p([a, b, cp]) = [ap, b, c]$ is a bijection*;

(c) *if $G \in G(d)$ and $K \in G$ then $\phi_p(K) \in G_p$.*

Proof. (a) We can choose $(a, b, c')$ in $K$ with $p \nmid a$. If $p = 2$ then, as $2 \mid d$ and $2 \nmid f$, we see that $2 \mid b$ and $d \equiv 8$ or $12 \pmod{16}$. If $c' \equiv 2 \pmod 4$ we take $c' = 2c$ and we are done. If $c' \not\equiv 2 \pmod 4$, from $d = b^2 - 4ac'$, we deduce that $c' \equiv 1 \pmod 2$ and $a + b + c' \equiv 2 \pmod 4$. Replacing $(a, b, c')$ by the equivalent form $(a, b + 2a, a + b + c')$, we obtain a form of the required type.

If $p \neq 2$ then $p \parallel d$. Choose $t$ such that $b' = 2at + b \equiv 0 \pmod p$, and set $c = (at^2 + bt + c')/p$. Then $(a, b', pc)$ is a form of the required type ($p \nmid c$, as $p \parallel d$ and $p \mid b'$) equivalent to $(a, b, c')$.

(b) The discriminant of $(ap, b, c)$ is $d$. It is easily checked that $(ap, b, c)$ is primitive. Hence $[ap, b, c] \in H(d)$. Next we show that $\phi_p$ is well-defined. Suppose that

$$[a, b, cp] = [a', b', c'p], \quad p \nmid aca'c',\ p \mid b,\ p \mid b'.$$

Thus there exist integers $\alpha,\beta,\gamma,\delta$ with $\alpha\delta-\beta\gamma=1$ and

$$\begin{aligned} a' &= a\alpha^2+b\alpha\gamma+cp\gamma^2,\\ b' &= 2a\alpha\beta+b(\alpha\delta+\beta\gamma)+2cp\gamma\delta,\\ c'p &= a\beta^2+b\beta\delta+cp\delta^2. \end{aligned} \tag{7.1}$$

As $p\,|\,b$ we see that $p\,|\,a\beta^2$, so that $p\,|\,\beta$, say $\beta=\beta'p$. Set $\gamma'=p\gamma$, so that $\alpha\delta-\beta'\gamma'=1$ and (7.1) can be rewritten as

$$\begin{aligned} a'p &= ap\alpha^2+b\alpha\gamma'+c\gamma'^2,\\ b' &= 2ap\alpha\beta'+b(\alpha\delta+\beta'\gamma')+2c\gamma'\delta,\\ c' &= ap\beta'^2+b\beta'\delta+c\delta^2, \end{aligned}$$

showing that $[ap,b,c]=[a'p,b',c']$, and thus $\phi_p$ is well-defined. Further

$$\phi_p^2([a,b,cp])=\phi_p([ap,b,c])=\phi_p([c,-b,ap])=[cp,-b,a]=[a,b,cp],$$

so that $\phi_p$ is an involution on $H(d)$, and thus a bijection.

(c) Let $G\in G(d)$ and $K=[a,b,cp]\in G$, where $p\nmid ac$ and $p\,|\,b$. Suppose that $\phi_p(K)$ belongs to the genus $\widetilde{G}$ of $G(d)$. We wish to show that $\widetilde{G}=G_p$.

Let $q^*\in P(d)$ with $p\nmid q^*$. Let $\mu$ be a positive integer coprime with $q^*$ which is represented by the form $(a,b,cp)\in K$. Clearly $p\mu$ is represented by the form $(ap,b,c)\in\phi_p(K)$. Then

$$\gamma_{q^*}(\widetilde{G})=\left(\frac{q^*}{p\mu}\right)=\left(\frac{q^*}{p}\right)\left(\frac{q^*}{\mu}\right)=\left(\frac{q^*}{p}\right)\gamma_{q^*}(G)=\gamma_{q^*}(G_p). \tag{7.2}$$

Now let $q^*\in P(d)$ be such that $p\,|\,q^*$. As $p\,|\,d$ and $p\nmid f$, there is only one such $q^*$, which we denote by $r^*$. Clearly $r^*\in P(\Delta)$. Hence

$$\begin{aligned} \gamma_{r^*}(\widetilde{G}) &= \prod_{\substack{q^*\in P(\Delta)\\ q^*\neq r^*}}\gamma_{q^*}(\widetilde{G}) && \text{(by Lemma 2.2)}\\ &= \prod_{\substack{q^*\in P(\Delta)\\ q^*\neq r^*}}\gamma_{q^*}(G_p) && \text{(by (7.2))}\\ &= \gamma_{r^*}(G_p). && \text{(by Lemma 2.2)} \end{aligned}$$

Thus we have shown that

$$\gamma_{q^*}(\widetilde{G})=\gamma_{q^*}(G_p)\quad\text{ for all } q^*\in P(d),$$

and so $\widetilde{G}=G_p$. ∎

Lemma 7.2. *Let $p$ be a prime with $p\,|\,n$, $p\,|\,d$ and $p\nmid f$. Then, for $K\in H(d)$, we have $R_K(n,d)=R_{\phi_p(K)}(n/p,d)$.*

Proof. We choose a form $(a,b,cp) \in K$ with $p \nmid ac$, $p \mid b$. Then $(ap,b,c) \in \phi_p(K)$. Set

$$S = \{(x,y) \in \mathbb{Z}^2 : ax^2 + bxy + cpy^2 = n\},$$
$$T = \{(X,Y) \in \mathbb{Z}^2 : apX^2 + bXY + cY^2 = n/p\}.$$

It is easy to check that $(X,Y) \to (pX,Y)$ defines a bijection from $T$ to $S$. ∎

Lemma 7.3. *Let $p$ be a prime with $p \mid n$, $p \mid d$ and $p \nmid f$. Then, for $G \in G(d)$, we have $R_G(n,d) = R_{G_p}(n/p,d)$.*

Proof. We have

$$\begin{aligned} R_G(n,d) &= \sum_{K \in G} R_K(n,d) \\ &= \sum_{K \in G} R_{\phi_p(K)}(n/p,d) \qquad \text{(by Lemma 7.2)} \\ &= \sum_{K' \in G_p} R_{K'}(n/p,d) \quad \text{(by Lemma 7.1(b), (c))} \\ &= R_{G_p}(n/p,d). \end{aligned}$$ ∎

Proof of Theorem 7.1. This theorem follows from Lemma 7.3 by applying it to all primes $p$ dividing $U$ taking multiplicity into account. ∎

**8. Formula for $R_G(n,d)$.** We now apply our two reduction formulae (Theorems 6.1 and 7.1) to obtain an explicit formula for $R_G(n,d)$.

Theorem 8.1. *Let $G \in G(d)$. If $\mathrm{Null}(n,d) = \emptyset$, then*

$$R_G(n,d) = \frac{w(d/M^2)}{2^{t(d)+1}} \cdot \frac{h(d)}{h(d/M^2)} \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G) S(n/M^2, d_1, d/M^2 d_1).$$

*If $\mathrm{Null}(n,d) \neq \emptyset$, then $R_G(n,d) = 0$.*

We begin by recalling Dirichlet's formula, see [5, p. 229], [4, p. 78].

Theorem 8.2 (Dirichlet). *If $(n,d) = 1$, then*

$$N(n,d) = w(d) \sum_{\nu \mid n} \left(\frac{d}{\nu}\right).$$

The next theorem is a consequence of Theorem 8.2.

Theorem 8.3. *If $(n,d) = 1$ and $G \in G(d)$, then*

$$R_G(n,d) = \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1).$$

Proof. If $N(n,d) > 0$, then $n$ is represented by at least one class in $H(d)$. Let $\widetilde{G}$ be a genus containing a class which represents $n$. As $(n,d)=1$ we have $(n,q^*)=1$ for all $q^* \in P(d)$. Thus

$$\gamma_{q^*}(\widetilde{G}) = \left(\frac{q^*}{n}\right) \quad \text{for all } q^* \in P(d),$$

and so $\widetilde{G}$ is unique. Hence

$$R_G(n,d) = \begin{cases} N(n,d) & \text{if } G = \widetilde{G}, \\ 0 & \text{if } G \neq \widetilde{G}, \end{cases}$$

that is,

$$\text{(8.1)} \qquad R_G(n,d) = \prod_{q^* \in P(d)} \frac{1}{2}\left(1 + \gamma_{q^*}(G)\left(\frac{q^*}{n}\right)\right) N(n,d).$$

The formula (8.1) trivially holds if $N(n,d)=0$.

By Theorem 8.2 and (8.1), we have

$$\begin{aligned} R_G(n,d) &= w(d) \prod_{q^* \in P(d)} \frac{1}{2}\left(1 + \gamma_{q^*}(G)\left(\frac{q^*}{n}\right)\right) \sum_{\nu \mid n} \left(\frac{d}{\nu}\right) \\ &= \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \left(\frac{d_1}{n}\right) \sum_{\mu\nu = n} \left(\frac{d}{\nu}\right), \end{aligned}$$

where in the case $d \equiv 0 \pmod{32}$ each term in the development of

$$\prod_{q^* \in P(d)} \left(1 + \gamma_{q^*}(G)\left(\frac{q^*}{n}\right)\right)$$

(recall $|P(d)| = t(d) + 2$) is obtained exactly twice in view of the relation $\gamma_{-4}(G)\gamma_8(G)\gamma_{-8}(G) = 1$. Hence

$$\begin{aligned} R_G(n,d) &= \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \sum_{\mu\nu = n} \left(\frac{d_1}{\mu}\right)\left(\frac{d_1}{\nu}\right)\left(\frac{d_1}{\nu}\right)\left(\frac{d/d_1}{\nu}\right) \\ &= \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \sum_{\mu\nu = n} \left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right) \\ &= \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1). \ \blacksquare \end{aligned}$$

Proof of Theorem 8.1. We have

$$
\begin{aligned}
R_G(n,d)
&= \frac{1}{2^{t(d)-t(d/M^2)}}\cdot\frac{h(d)}{h(d/M^2)}R_{G_M}(n/M^2,d/M^2) \qquad \text{(by Theorem 6.1)}\\
&= \frac{1}{2^{t(d)-t(d/M^2)}}\cdot\frac{h(d)}{h(d/M^2)}R_{G_{MQ}}(n/M^2Q,d/M^2) \quad \text{(by Theorem 7.1)}
\end{aligned}
$$

as $U(n/M^2,d/M^2)=Q$ (by (4.3)). By Lemma 4.1(a) we have, as $\mathrm{Null}(n,d) = \emptyset$,

$$\left(\frac{n}{M^2Q},\frac{d}{M^2}\right)=1 \quad\text{and}\quad \left(\frac{n}{M^2},\frac{f}{M}\right)=1,$$

so that, by Theorem 8.3, we have

$$
\begin{aligned}
&R_{G_{MQ}}\left(\frac{n}{M^2Q},\frac{d}{M^2}\right)\\
&\qquad= \frac{w(d/M^2)}{2^{t(d/M^2)+1}}\sum_{d_1\in F(d/M^2)}\gamma_{d_1}(G_{MQ})S\left(\frac{n}{M^2Q},d_1,\frac{d}{M^2d_1}\right),
\end{aligned}
$$

and thus, by Lemma 5.2, we obtain

$$R_{G_{MQ}}\left(\frac{n}{M^2Q},\frac{d}{M^2}\right)=\frac{w(d/M^2)}{2^{t(d/M^2)+1}}\sum_{d_1\in F(d/M^2)}\gamma_{d_1}(G_M)S\left(\frac{n}{M^2},d_1,\frac{d}{M^2d_1}\right).$$

Further, by Lemma 3.1(b), we have $\gamma_{d_1}(G_M)=\gamma_{d_1}(G)$, and the result follows. ∎

Corollary 8.1. *If $d$ is fundamental, then*

$$R_G(n,d)=\frac{w(d)}{2^{t(d)+1}}\sum_{d_1\in F(d)}\gamma_{d_1}(G)\sum_{\mu\nu=n}\left(\frac{d_1}{\mu}\right)\left(\frac{d/d_1}{\nu}\right).$$

Proof. As $d$ is fundamental, we have $f=1$, $M=1$, and $\mathrm{Null}(n,d)=\emptyset$, and the result follows immediately from Theorem 8.1. ∎

**9. Determination of $N(n,d)$.** We now use Theorem 8.1 to obtain a formula for $N(n,d)$, which generalizes Dirichlet's formula (Theorem 8.2). Special cases of Theorem 9.1 are given in Hardy and Williams [7, pp. 104–105] and Schinzel and Zannier [11, Lemma 4, p. 48].

Theorem 9.1. *If* $\mathrm{Null}(n,d)=\emptyset$ *then*

$$N(n,d)=w(d/M^2)\frac{h(d)}{h(d/M^2)}\sum_{\nu\mid n/M^2}\left(\frac{\Delta}{\nu}\right).$$

*If* $\mathrm{Null}(n,d)\neq\emptyset$ *then* $N(n,d)=0$.

Proof. If $\mathrm{Null}(n,d) \neq \emptyset$, we have $N(n,d) = 0$ by Proposition 4.1. If $\mathrm{Null}(n,d) = \emptyset$ then

$$\begin{aligned}
N(n,d) &= \sum_{G\in G(d)} R_G(n,d) \qquad \text{(by (1.11))}\\
&= \sum_{G\in G(d)} \frac{w(d/M^2)}{2^{t(d)+1}}\cdot\frac{h(d)}{h(d/M^2)}\\
&\quad\times \sum_{d_1\in F(d/M^2)} \gamma_{d_1}(G) S(n/M^2, d_1, d/M^2 d_1) \qquad \text{(by Theorem 8.1)}\\
&= \frac{w(d/M^2)}{2^{t(d)+1}}\cdot\frac{h(d)}{h(d/M^2)}\\
&\quad\times \sum_{d_1\in F(d/M^2)} \Big\{\sum_{G\in G(d)} \gamma_{d_1}(G)\Big\} S(n/M^2, d_1, d/M^2 d_1)\\
&= \frac{w(d/M^2)}{2^{t(d)+1}}\cdot\frac{h(d)}{h(d/M^2)}\\
&\quad\times \{2^{t(d)} S(n/M^2, 1, d/M^2) + 2^{t(d)} S(n/M^2, \Delta, d/M^2\Delta)\}\\
&\qquad \text{(by (2.13))}\\
&= \frac{w(d/M^2)}{2}\cdot\frac{h(d)}{h(d/M^2)}\\
&\quad\times \{S(n/M^2, 1, d/M^2) + S(n/M^2, \Delta, d/M^2\Delta)\}\\
&= \frac{w(d/M^2)}{2}\cdot\frac{h(d)}{h(d/M^2)}\\
&\quad\times \sum_{\mu\nu=n/M^2} \Big\{\Big(\frac{1}{\mu}\Big)\Big(\frac{\Delta(f/M)^2}{\nu}\Big) + \Big(\frac{\Delta}{\mu}\Big)\Big(\frac{(f/M)^2}{\nu}\Big)\Big\}\\
&\qquad \text{(by (5.1))}\\
&= \frac{w(d/M^2)}{2}\cdot\frac{h(d)}{h(d/M^2)} \sum_{\mu\nu=n/M^2} \Big\{\Big(\frac{\Delta}{\nu}\Big) + \Big(\frac{\Delta}{\mu}\Big)\Big\},
\end{aligned}$$

as $(n/M^2, f/M) = 1$ by Lemma 4.1(a), and Theorem 9.1 follows. ∎

Our next result gives upper bounds for $N(n,d)$. To prove these we need the following two inequalities. Let $\tau(n)$ denote the number of divisors of $n$. Then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\tau(n) \le C(\varepsilon) n^{\varepsilon} \tag{9.1}$$

(see [14, Corollary 1.1, p. 92]). Also there exists a constant $C_1 > 0$ such that

$$\prod_{p\mid n}\left(1+\frac{1}{p}\right) \le C_1 \log\log n, \qquad n \ge 3 \tag{9.2}$$

(see [14, p. 98, formula (19)]).

We also make use here and in the next section of Gauss's formula

$$\frac{h(Dk^2)}{h(D)} = \frac{w(Dk^2)}{w(D)} k \prod_{p\mid k}\left(1-\frac{\left(\frac{D}{p}\right)}{p}\right), \tag{9.3}$$

where $D$ is a negative discriminant, and $k$ is a positive integer (see for example [3, p. 217]).

COROLLARY 9.1. (a) *For any $\varepsilon > 0$ there exists a constant $C_2(\varepsilon) > 0$ such that*

$$0 \le N(n,d) \le C_2(\varepsilon)\, f\, n^{\varepsilon}.$$

(b) *There exists a constant $C_3 > 0$ such that*

$$0 \le N(n,d) \le \begin{cases} 12 & \text{if } n = 1,2,\\ C_3 n^{1/2}\log\log n & \text{if } n \ge 3.\end{cases}$$

Proof. If $\mathrm{Null}(n,d) \ne \emptyset$ then $N(n,d) = 0$ and the assertions are trivial. Thus we may suppose that $\mathrm{Null}(n,d) = \emptyset$. As

$$w(d/M^2)\frac{h(d)}{h(d/M^2)} = w(d)M\prod_{p\mid M}\left(1-\frac{\left(\frac{d/M^2}{p}\right)}{p}\right) \qquad \text{(by (9.3))}$$

$$\le 6M\prod_{p\mid M}\left(1+\frac{1}{p}\right) \le 6M\prod_{p\mid n}\left(1+\frac{1}{p}\right)$$

and

$$\sum_{\nu\mid n/M^2}\left(\frac{\Delta}{\nu}\right) \le \sum_{\nu\mid n/M^2} 1 \le \tau(n/M^2),$$

we have, by Theorem 9.1,

$$0 \le N(n,d) \le 6M\prod_{p\mid n}\left(1+\frac{1}{p}\right)\tau(n/M^2).$$

To prove (a) we use the inequalities

$$M \le f, \qquad \prod_{p\mid n}\left(1+\frac{1}{p}\right) \le \tau(n) \le C(\varepsilon/2)n^{\varepsilon/2},$$

$$\tau(n/M^2) \le \tau(n) \le C(\varepsilon/2)n^{\varepsilon/2},$$

where $\varepsilon > 0$. To prove (b) we use the inequalities

$$M\tau(n/M^2) \le MC(1/2)\left(\frac{n}{M^2}\right)^{1/2} = C(1/2)n^{1/2},$$

$$\prod_{p\mid n}\left(1+\frac{1}{p}\right) \le C_1 \log\log n, \qquad n \ge 3,$$

and

$$N(1,d) = w(d) \le 6, \qquad N(2,d) = w(d)\left\{1+\left(\frac{\Delta}{2}\right)\right\} \le 12. \ \blacksquare$$

Remark 9.1. If $(n,f)=1$, Theorem 9.1 reduces to

$$N(n,d) = w(d)\sum_{\nu\mid n}\left(\frac{\Delta}{\nu}\right) = w(d)\sum_{\nu\mid n}\left(\frac{d}{\nu}\right). \tag{9.4}$$

This is a generalization of Dirichlet's formula (Theorem 8.2).

If $d$ is a fundamental discriminant, then $f=1$, and (9.4) holds for all $n$. This result appears to be known but not well-known.

**10. Evaluation of the Dirichlet series $\sum_{n=1}^{\infty} R_G(n,d)/n^s$.** Let $D$ be a discriminant. For $s>1$ the Dirichlet $L$-series is given by

$$L(s,D) = \sum_{n=1}^{\infty}\frac{\left(\frac{D}{n}\right)}{n^s},$$

where $\left(\frac{D}{n}\right)$ is the Kronecker symbol defined in (2.1)–(2.3). In particular, $L(s,1)=\sum_{n=1}^{\infty} 1/n^s = \zeta(s)$, the Riemann zeta function. Also,

$$\begin{aligned} L(s,D) &= \sum_{n=1}^{\infty}\frac{\left(\frac{\Delta(D)(f(D))^2}{n}\right)}{n^s} = \sum_{\substack{n=1\\(n,f(D))=1}}^{\infty}\frac{\left(\frac{\Delta(D)}{n}\right)}{n^s} \\ &= \prod_{p\mid f(D)}\left(1-\frac{\left(\frac{\Delta(D)}{p}\right)}{p^s}\right)\sum_{n=1}^{\infty}\frac{\left(\frac{\Delta(D)}{n}\right)}{n^s} \\ &= \prod_{p\mid f(D)}\left(1-\frac{\left(\frac{\Delta(D)}{p}\right)}{p^s}\right)L(s,\Delta(D)). \end{aligned}$$

By Corollary 9.1(a) we have, for any $\varepsilon>0$,

$$0 \le R_G(n,d) \le N(n,d) \le C_2(\varepsilon) f n^{\varepsilon},$$

so that $\sum_{n=1}^{\infty} R_G(n,d)/n^s$ converges absolutely for $s>1$ and uniformly for $s \ge 1+\varepsilon$.

THEOREM 10.1. *Let $G \in G(d)$. For $s > 1$ we have*

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \frac{h(d)}{2^{t(d)}} \sum_{m\mid f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^{2s}} \times \sum_{\substack{d_1 \in F(d/m^2)\\ d_1>0}} \gamma_{d_1}(G) \prod_{p\mid f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p^s}\right)\left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right) \times L(s,d_1)L(s,\Delta(d/d_1)).$$

Proof. For $s > 1$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} &= \sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset}}^{\infty} \frac{R_G(n,d)}{n^s} \\
&= \sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset}}^{\infty} \frac{1}{n^s} \cdot \frac{w(d/M(n,d)^2)}{2^{t(d)+1}} \cdot \frac{h(d)}{h(d/M(n,d)^2)} \\
&\quad \times \sum_{d_1 \in F(d/M(n,d)^2)} \gamma_{d_1}(G) S(n/M(n,d)^2, d_1, \Delta(d/d_1)) \quad \text{(by Theorem 8.1)} \\
&= \frac{h(d)}{2^{t(d)+1}} \sum_{m\mid f} \frac{w(d/m^2)}{h(d/m^2)} \sum_{d_1 \in F(d/m^2)} \gamma_{d_1}(G) \times \sum_{\substack{n=1\\ \mathrm{Null}(n,d)=\emptyset\\ M(n,d)=m}}^{\infty} \frac{S(n/m^2, d_1, \Delta(d/d_1))}{n^s} \\
&= \frac{h(d)}{2^{t(d)+1}} \sum_{m\mid f} \frac{w(d/m^2)}{h(d/m^2)} \sum_{d_1 \in F(d/m^2)} \gamma_{d_1}(G) \\
&\quad \times \sum_{\substack{n=1\\ (n/m^2, f/m)=1}}^{\infty} \frac{S(n/m^2, d_1, \Delta(d/d_1))}{n^s} \quad \text{(by Lemma 4.1(b))} \\
&= \frac{h(d)}{2^{t(d)+1}} \sum_{m\mid f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^{2s}} \sum_{d_1 \in F(d/m^2)} \gamma_{d_1}(G) \times \sum_{\substack{N=1\\ (N, f/m)=1}}^{\infty} \frac{S(N, d_1, \Delta(d/d_1))}{N^s}
\end{aligned}$$

$$= \frac{h(d)}{2^{t(d)+1}} \sum_{m|f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^{2s}} \sum_{d_1 \in F(d/m^2)} \gamma_{d_1}(G)$$

$$\times \sum_{\substack{\mu=1 \\ (\mu,f/m)=1}}^{\infty} \frac{\left(\frac{d_1}{\mu}\right)}{\mu^s} \sum_{\substack{\nu=1 \\ (\nu,f/m)=1}}^{\infty} \frac{\left(\frac{\Delta(d/d_1)}{\nu}\right)}{\nu^s}$$

$$= \frac{h(d)}{2^{t(d)+1}} \sum_{m|f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^{2s}} \sum_{d_1 \in F(d/m^2)} \gamma_{d_1}(G)$$

$$\times \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p^s}\right) L(s, d_1) \prod_{p|f/m} \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p^s}\right) L(s, \Delta(d/d_1)).$$

The assertion of the theorem now follows by noting that $d_1 \to \Delta(d/d_1)$ is a bijection on $F(d/m^2)$ (by (2.11)), $\gamma_{d_1}(G) = \gamma_{\Delta(d/d_1)}(G)$ (by (2.12)), and $d_1 \Delta(d/d_1) < 0$. ■

THEOREM 10.2. *Let $G \in G(d)$. For $s > 1$ we have*

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \frac{\pi h(d)}{2^{t(d)-1}\sqrt{|d|}} \cdot \frac{1}{s-1} + B_G(d) + O(s-1),$$

*where*

$$B_G(d) = \frac{\pi h(d)}{2^{t(d)-1}\sqrt{|d|}} \log 2\pi + \frac{\pi \gamma h(d)}{2^{t(d)-2}\sqrt{|d|}}$$

$$- \frac{\pi}{2^{t(d)-1}} \cdot \frac{h(d)}{\sqrt{|d|}} \sum_{p|f} \alpha_p(\Delta, f) \log p$$

$$- \frac{\pi}{2^{t(d)}} \cdot \frac{h(d) w(\Delta)}{\sqrt{|d|} h(\Delta)} \sum_{m=1}^{|\Delta|} \left(\frac{\Delta}{m}\right) \log \Gamma\left(\frac{m}{|\Delta|}\right)$$

$$- \frac{8\pi}{\sqrt{|d|}} \sum_{\substack{d_1 \in F(d) \\ d_1 > 1}} \beta(d_1, d, G) \log \varepsilon_{d_1},$$

*where $\alpha_p(\Delta, f)$ and $\beta(d_1, d, G)$ are defined in Section 1, and $\gamma$ denotes Euler's constant.*

Proof. For $s > 1$, by Theorem 10.1, we have

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = S_1 + S_2, \tag{10.1}$$

where

$$S_1 = \frac{h(d)}{2^{t(d)}} \sum_{m|f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^{2s}} \times \prod_{p|f/m} \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p^s}\right)\zeta(s)L(s,\Delta) \tag{10.2}$$

and

$$S_2 = \frac{h(d)}{2^{t(d)}} \sum_{m|f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^{2s}} \sum_{\substack{d_1 \in F(d/m^2) \\ d_1 > 1}} \gamma_{d_1}(G) \times \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p^s}\right)\left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p^s}\right) \times L(s,d_1)L(s,\Delta(d/d_1)). \tag{10.3}$$

We treat $S_2$ first. We have

$$S_2 = \frac{h(d)}{2^{t(d)}} \sum_{m|f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^{2s}} \sum_{\substack{d_1 \in F(d/m^2) \\ d_1 > 1}} \gamma_{d_1}(G) \times \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right)\left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right) \times L(1,d_1)L(1,\Delta(d/d_1)) + O(s-1).$$

By Dirichlet's classnumber formulae (see for example [3, p. 171])

$$L(1,d_1) = \frac{2h(d_1)\log\varepsilon_{d_1}}{\sqrt{d_1}},$$

$$L(1,\Delta(d/d_1)) = \frac{2\pi h(\Delta(d/d_1))}{w(\Delta(d/d_1))\sqrt{|\Delta(d/d_1)|}},$$

we obtain

$$S_2 = \frac{\pi h(d)}{2^{t(d)-2}} \sum_{m|f} \frac{w(d/m^2)}{h(d/m^2)} \cdot \frac{1}{m^2} \sum_{\substack{d_1 \in F(d/m^2) \\ d_1 > 1}} \frac{\gamma_{d_1}(G)h(d_1)h(\Delta(d/d_1))\log\varepsilon_{d_1}}{w(\Delta(d/d_1))\sqrt{d_1|\Delta(d/d_1)|}} \times \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right)\left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right) + O(s-1).$$

Next, appealing to (9.3), we deduce

$$S_2 = \frac{\pi w(d)}{2^{t(d)-2}} \sum_{m|f} \frac{1}{m} \sum_{\substack{d_1 \in F(d/m^2) \\ d_1 > 1}} \frac{\gamma_{d_1}(G) h(d_1) h(\Delta(d/d_1)) \log \varepsilon_{d_1}}{w(\Delta(d/d_1)) \sqrt{d_1 |\Delta(d/d_1)|}}$$
$$\times \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p}\right) \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right)\left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right) + O(s-1).$$

Then, interchanging the order of summation, and using $|d|/d_1 = |\Delta(d/d_1)| \times f(d/d_1)^2$, we deduce

$$S_2 = -\frac{8\pi}{\sqrt{|d|}} \sum_{\substack{d_1 \in F(d) \\ d_1 > 1}} \beta(d_1, d, G) \log \varepsilon_{d_1} + O(s-1). \tag{10.4}$$

Now we turn to the determination of $S_1$. As $s > 1$, we have

$$\frac{1}{m^{2s}} = \frac{1}{m^2} - 2(s-1)\frac{\log m}{m^2} + O((s-1)^2),$$
$$\prod_{p|f/m} \left(1 - \frac{1}{p^s}\right)$$
$$= \prod_{p|f/m} \left(1 - \frac{1}{p}\right)\left\{1 + (s-1) \sum_{p|f/m} \frac{\log p}{p-1} + O((s-1)^2)\right\},$$
$$\prod_{p|f/m} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p^s}\right)$$
$$= \prod_{p|f/m} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p}\right)\left\{1 + (s-1) \sum_{p|f/M} \frac{\left(\frac{\Delta}{p}\right)\log p}{p - \left(\frac{\Delta}{p}\right)} + O((s-1)^2)\right\},$$
$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1),$$
$$L(s, \Delta) = L(1, \Delta) + (s-1)L'(1, \Delta) + O((s-1)^2).$$

Also, from [3, p. 171] and [12, p. 110], we have

$$L(1, \Delta) = \frac{2\pi h(\Delta)}{w(\Delta)\sqrt{|\Delta|}},$$
$$L'(1, \Delta) = -\frac{\pi}{\sqrt{|\Delta|}} \sum_{m=1}^{|\Delta|} \left(\frac{\Delta}{m}\right) \log \Gamma\left(\frac{m}{|\Delta|}\right) + \frac{2h(\Delta)\pi(\gamma + \log 2\pi)}{w(\Delta)\sqrt{|\Delta|}}.$$

Using these results in (10.2), together with (9.3) and the relation

$$\sum_{m|f} \frac{1}{m} \prod_{p|f/m} \left(1 - \frac{1}{p}\right) = 1,$$

we obtain after a long but straightforward calculation

$$(10.5) \qquad S_1 = \frac{\pi h(d)}{2^{t(d)}\sqrt{|d|}} \left\{ \frac{2}{s-1} + (2\log 2\pi + 4\gamma) - 2\sum_{m|f} \frac{1}{m} \prod_{p|f/m} \left(1 - \frac{1}{p}\right) \right.$$
$$\times \left\{ 2\log m - \sum_{p|f/m} \left( \frac{1}{p-1} + \frac{\left(\frac{\Delta}{p}\right)}{p - \left(\frac{\Delta}{p}\right)} \right) \log p \right\}$$
$$\left. - \frac{w(\Delta)}{h(\Delta)} \sum_{m=1}^{|\Delta|} \left(\frac{\Delta}{m}\right) \log \Gamma\left(\frac{m}{|\Delta|}\right) \right\}.$$

Next it is easy to check that

$$A(f) = \sum_{m|f} \frac{1}{m} \prod_{p|f/m} \left(1 - \frac{1}{p}\right) \log m$$

is an additive function of $f$. Using this we deduce that

$$(10.6) \qquad \sum_{m|f} \frac{1}{m} \prod_{p|f/m} \left(1 - \frac{1}{p}\right) \log m = \sum_{p|f} \frac{p^{v_p(f)} - 1}{p^{v_p(f)}(p-1)} \log p.$$

An easy calculation shows that

$$(10.7) \qquad \sum_{m|f} \frac{1}{m} \prod_{p|f/m} \left(1 - \frac{1}{p}\right) \sum_{p|f/m} \left( \frac{1}{p-1} + \frac{\left(\frac{\Delta}{p}\right)}{p - \left(\frac{\Delta}{p}\right)} \right) \log p$$
$$= \sum_{p|f} \frac{p^{v_p(f)} - 1}{p^{v_p(f)}} \left( \frac{1}{p-1} + \frac{\left(\frac{\Delta}{p}\right)}{p - \left(\frac{\Delta}{p}\right)} \right) \log p.$$

From (10.6) and (10.7), we deduce that

$$(10.8) \qquad \sum_{m|f} \frac{1}{m} \prod_{p|f/m} \left(1 - \frac{1}{p}\right) \left\{ 2\log m - \sum_{p|f/m} \left( \frac{1}{p-1} + \frac{\left(\frac{\Delta}{p}\right)}{p - \left(\frac{\Delta}{p}\right)} \right) \log p \right\}$$
$$= \sum_{p|f} \alpha_p(\Delta, f) \log p.$$

The theorem now follows from (10.1), (10.4), (10.5), and (10.8). ∎

Finally, we give the proof of our main theorem, using the approach of Chowla and Selberg [12].

Proof of Theorem 1.1. Kronecker's "Grenz-Formel" (see for example [13, Theorem 1, p. 14]) asserts that for $s > 1$ we have

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(am^2+bmn+cn^2)^s} = \frac{2\pi}{\sqrt{|d|}} \cdot \frac{1}{s-1} + K(a,b,c) + O(s-1), \tag{10.9}$$

where

$$K(a,b,c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi \log|d|}{\sqrt{|d|}} + \frac{2\pi}{\sqrt{|d|}} \log a - \frac{8\pi}{\sqrt{|d|}} \log\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|. \tag{10.10}$$

Thus

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \sum_{[a,b,c]\in G} \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(am^2+bmn+cn^2)^s} = \frac{\pi h(d)}{2^{t(d)-1}\sqrt{|d|}} \cdot \frac{1}{s-1} + \sum_{[a,b,c]\in G} K(a,b,c) + O(s-1). \tag{10.11}$$

From (10.11) and Theorem 10.2, we deduce that

$$\sum_{[a,b,c]\in G} K(a,b,c) = B_G(d). \tag{10.12}$$

Using the expressions for $K(a,b,c)$ (eqn. (10.10)) and $B_G(d)$ (Theorem 10.2) in (10.12), and exponentiating, we obtain the assertion of Theorem 1.1. ∎

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DEPARTMENT OF MATHEMATICS
CANISIUS COLLEGE
BUFFALO, NEW YORK 14208
U.S.A.

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE NANCY 1
B.P. 239
54506 VANDOEUVRE LES NANCY CEDEX, FRANCE

CENTRE FOR RESEARCH IN ALGEBRA AND NUMBER THEORY
DEPARTMENT OF MATHEMATICS AND STATISTICS
CARLETON UNIVERSITY
OTTAWA, ONTARIO, CANADA KIS 5B6


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