Jour. of Math. Sc.
Vol. 6, No. 1 (1995) 15—18

THE SUBFIELDS OF THE SPLITTING FIELD OF A SOLVABLE QUINTIC TRINOMIAL X^5+aX+b

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Let Q denote the field of rational numbers, and set $Q^* = Q\setminus\{0\}$. Let $a \in Q^*$ and $b \in Q^*$ be such that the quintic trinomial $f(X) = X^5+aX+b$ is both irreducible and solvable. Polynomials of this type are characterized in [3, Theorem]. Let L denote the splitting field of f. Let r denote the unique retional root of the resolvent sextic of X^5+aX+b [3, eqn. (17)], and set

$$c = \left|\frac{3r-16a}{4r+12a}\right|, \quad \epsilon = \operatorname{sgn}\left(\frac{3r-16a}{4r+12a}\right), \quad e = \frac{-5b\epsilon}{2r+4a}, \tag{1}$$

so that c is a nonnegative rational number, $\epsilon = \pm 1$, and e is a nonzero rational number. It is shown in [3] that

$$a = \frac{5e^4(3-4\epsilon c)}{c^2+1}, \quad b = \frac{-4e^5(11\epsilon+2c)}{c^2+1}. \tag{2}$$

The Galois group G_f of f is the dihedral group D_5 of order 10 if $5(c^2+1) \in Q^2$, and is the Frobenius group F_{20} of order 20 if $5(c^2+1) \notin Q^2$.

If $G_f = D_5$ then G_f has five subgroups of order 2, and one of order 5. The five quintic subfields of L are $Q(\theta_i)$, $i = 1,2,3,4,5$, where

* Research supported by Natural sciences and Engineering Research Council of Canada Grant A-7233.

$$\theta_i = e(w^i u_1 + w^{2i} u_2 + w^{3i} u_3 + w^{4i} u_4),$$
$$w = \exp(2\pi i/5),$$
$$u_1 = \left(\frac{v_1^2 v_3}{D^2}\right)^{1/5}, \; u_2 = \left(\frac{v_3^2 v_4}{D^2}\right)^{1/5}, \; u_3 = \left(\frac{v_2^2 v_1}{D_2}\right)^{1/5}, \; v_4 = \left(\frac{v_4^2 v_2}{D^2}\right)^{1/5},$$
$$v_1 = \sqrt{D} + \sqrt{D - \epsilon\sqrt{D}}, \quad v_2 = -\sqrt{D} - \sqrt{D + \epsilon\sqrt{D}},$$
$$v_3 = \sqrt{D} + \sqrt{D + \epsilon\sqrt{D}}, \quad v_4 = \sqrt{D} - \sqrt{D - \epsilon\sqrt{D}},$$
$$D = c^2 + 1,$$

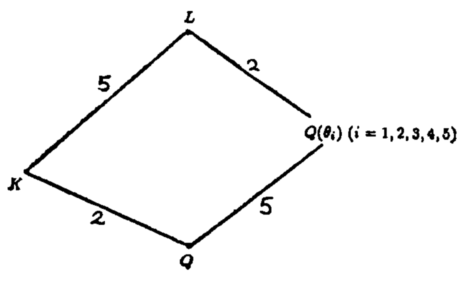
see [3, Theorem]. It remains to determine the unique quadratic subfield K of L. This is done in the theorem below making use of the work of Dummit [1].

If $G_f = F_{20}$ then G_f has five subgraphs of order 2, five subgroups of order 4, one of order 5, and one of order 10. The unique quadratic subfield of L is

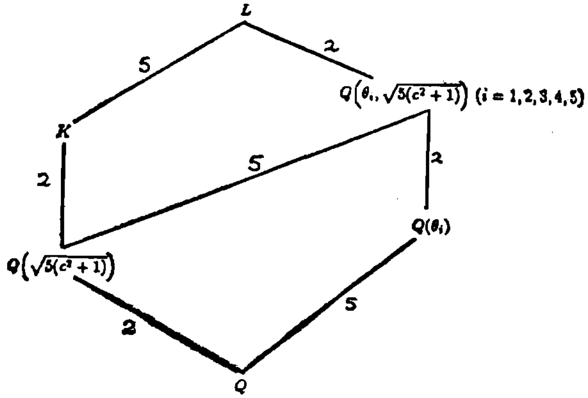
$$Q(\sqrt{\mathrm{disc}(f)}) = Q(\sqrt{4^4 a^5 + 5^5 b^4}) = Q(\sqrt{5(c^2+1)}),$$

see [2, eqn. (28)]. The five quintic subfields of L are $Q(\theta_i)$, $i = 1,2,3,4,5$, where θ_i is given above, and the five subfields of L of order 10 are $Q(\theta_i, \sqrt{5(c^2+1)})$. It remains to determine the unique quartic subfield K of L. This field is cyclic, and is given in the theorem below.

$G_f = D_5$



$G_f = F_{20}$



Theorem Let $f(X) = X^5 + aX + b \in Q[X]$ be a solvable, irreducible quintic trinomial with $ab \neq 0$. Define c, ϵ and e as in (1). Let L denote the splitting field of f, and let G_f denote the Galois group of f. Let

$$K = \begin{cases} \text{unique quadratic subfield of L when } G_f = D_5, \\ \text{unique (cyclic) quartic subfield of L when } G_f = F_{20}. \end{cases}$$

Then

$$K = Q\left(\sqrt{-5-(1+2\epsilon c)\sqrt{\frac{5}{c^2+1}}}\right).$$

Proof By [1, Theorem 2], we have

$$K = Q(\sqrt{(T_1+T_2\Delta)^2 - 4(T_3+T_4\Delta)}) = Q(\sqrt{(T_1-T_2\Delta)^2-4(T_3-T_4\Delta)}),$$

where T_1, T_2, T_3, T_4 are defined in (8.1'), (8.2'), (8.3'), (8.4') of [1] respectively, and $\Delta^2 = 4^4a^5+5^5b^4$. Each T_i is a rational function of a, b, and r. From (1) and (2) we see that

$$r = \frac{20e^4(4+3\epsilon c)}{c^2+1}$$

Since the splitting fields of X^5+aX+b and $X^5+(a/e^4)X+(b/e^5)$ are exactly the same field L, we can take $e=1$ without loss of generality. Thus we have

$$a = \frac{5(3-4\epsilon c)}{c^2+1},\quad b = \frac{-4(11\epsilon+2c)}{c^2+1},\quad r = \frac{20(4+3\epsilon c)}{c^2+1},$$

and, putting these expressions into (8.1'), (8.2'), (8.3'), (8.4') of [1], we obtain the T_i's as functions of c and ϵ. Also, by [2, eq. (28)], we may choose

$$\Delta = \frac{2^4 5^2}{(c^2+1)^2}(4\epsilon c^3 - 84c^2 - 37\epsilon c - 122)\sqrt{\frac{5}{c^2+1}}.$$

Then, using MAPLE to perform the algebric calculations, we obtain

$$(T_1+T_2\Delta)^2 - 4(T_3+T_4\Delta)$$

$$= \frac{2^2 5^8}{(c^2+1)^2}\left((-25-20\epsilon c-40c^2)+(11+6\epsilon c+12c^2+8\epsilon c^3)\sqrt{\frac{5}{c^2+1}}\right),$$

showing that

$$K = Q\left(\sqrt{(-25-20\epsilon c-40c^2)+(11+6\epsilon c+12c^2+8\epsilon c^3)\sqrt{\frac{5}{c^2+1}}}\right).$$

If $\epsilon c = 2$ then $K = Q(\sqrt{-90}) = Q(\sqrt{-10}) = Q\left(\sqrt{-5-(1+2\epsilon c)\sqrt{\frac{5}{c^2+1}}}\right).$

If $\epsilon c \neq 2$ the equality

$$(-25-20\epsilon c-40c^2)-(11+6\epsilon c+12c^2+8\epsilon c^3)\sqrt{\frac{5}{c^2+1}}$$

$$=\left(\frac{6c^2+\epsilon c+4-2(\epsilon c+1)\sqrt{5(c^2+1)}}{\epsilon c-2}\right)^2\left(-5-(1+2\epsilon c)\sqrt{\frac{5}{c^2+1}}\right)$$

shows that

$$K=Q\left(\sqrt{-5-(1+2\epsilon c)\sqrt{\frac{5}{c^2+1}}}\right).$$

We close with a few examples.

X^5+aX+b	r	c	ϵ	e	G_t	K
$X^5-5X+12$	40	2	1	−1	D_5	$Q(\sqrt{-10})$
$X^5+11X+44$	88	2/11	1	−1	D_5	$Q(\sqrt{-2})$
$X^5+15X+12$	0	4/3	−1	1	F_{20}	$Q(\sqrt{-5+\sqrt{5}})$
$X^5-40X+64$	10	7	1	−2	F_{20}	$Q(\sqrt{-5-\frac{3}{2}\sqrt{10}})$
$X^5+15X+44$	80	0	1	−1	F_{20}	$Q(\sqrt{-5+\sqrt{5}})$
$X^5+20X+32$	40	1/2	−1	1	D_5	$Q(\sqrt{-5})$
$X^5+\frac{15}{2}X+2$	−10	3	−1	1	F_{20}	$Q(\sqrt{-5+\frac{5}{2}\sqrt{2}})$
$X^5-1900X-8800$	8200	11/2	1	5	D_5	$Q(\sqrt{-5})$

Jensen and Yui [2, Theorem II.3.6] have calculated the quadratic subfield K in certain cases when $G_t=D_5$.

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