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An elementary remark on the distribution of integers representable by a positive-definite integral binary quadratic form

By

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In this note we prove an elementary result concerning the distribution of the positive integers which are represented by a positive-definite integral binary quadratic form

Theorem 1. *Let $f(X, Y) = aX^2 + bXY + cY^2$ be a positive-definite integral binary quadratic form of discriminant $-\Delta\,(= b^2 - 4ac < 0)$. Let m_1 be the least positive integer represented by f. Then, for every integer $n \geqq m_1$, there exist integers x and y such that*

$$n < ax^2 + bxy + cy^2 < n + 2m_1^{1/4}\Delta^{1/4}n^{1/4} + m_1.$$

Proof. Replacing the form f by an equivalent form we may suppose that $m_1 = c$. We define integers x and y by

$$x = \left[\left(\frac{4cn}{\Delta}\right)^{1/2}\right], \quad y = \left[\frac{(4cn - \Delta x^2)^{1/2} - bx}{2c}\right] + 1.$$

Next we define real numbers ε and δ by

$$\varepsilon = \left\{\left(\frac{4cn}{\Delta}\right)^{1/2}\right\}, \quad \delta = \left\{\frac{(4cn - \Delta x^2)^{1/2} - bx}{2c}\right\},$$

where $\{\theta\} = \theta - [\theta]$ denotes the fractional part of the real number θ, so that

$$0 \leqq \varepsilon < 1, \quad 0 \leqq \delta < 1,$$

and

$$x = \left(\frac{4cn}{\Delta}\right)^{1/2} - \varepsilon, \quad y = \frac{(4cn - \Delta x^2)^{1/2} - bx}{2c} + (1 - \delta).$$

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Then we have

$$\begin{aligned} a x^2 + b x y + c y^2 &= a x^2 + \frac{1}{c}(c y)(b x + c y) \\ &= a x^2 + \frac{1}{c}\left(\frac{(4 c n - \Delta x^2)^{1/2} - b x}{2} + c(1-\delta)\right)\left(\frac{(4 c n - \Delta x^2)^{1/2} + b x}{2} + c(1-\delta)\right) \\ &= a x^2 + \frac{1}{c}\left(\frac{4 c n - \Delta x^2 - b^2 x^2}{4} + c(1-\delta)(4 c n - \Delta x^2)^{1/2} + c^2(1-\delta)^2\right) \\ &= n + (1-\delta)(4 c n - \Delta x^2)^{1/2} + c(1-\delta)^2 \quad (\text{as } \Delta = 4 a c - b^2) \\ &= n + (1-\delta)(4 \varepsilon c^{1/2} \Delta^{1/2} n^{1/2} - \Delta \varepsilon^2)^{1/2} + c(1-\delta)^2. \end{aligned}$$

Clearly we have

$$n < a x^2 + b x y + c y^2 < n + 2 c^{1/4} \Delta^{1/4} n^{1/4} + c,$$

as asserted. □

Corollary 1. *With the same notation and assumptions as in Theorem* 1, *for every integer* $n \geqq m_1$, *there exist integers* x *and* y *such that*

$$n < a x^2 + b x y + c y^2 < n + \frac{2}{3^{1/8}} \Delta^{3/8} n^{1/4} + \frac{\Delta^{1/2}}{3^{1/2}}.$$

Proof. This follows immediately from Theorem 1 and the well-known bound $m_1 \leqq \left(\frac{\Delta}{3}\right)^{1/2}$ [1: p. 30], [3: Theorem]. □

Our second theorem improves Theorem 1 in the case when $f(X, Y) = a X^2 + c Y^2$ and $n \geqq c^3/a^2$. This generalizes a theorem of Uchiyama [2: Theorem 1].

Theorem 2. *Let* $a X^2 + c Y^2$ *be a positive-definite integral binary quadratic form with* $a \leqq c$. *Then, for every integer* $n \geqq c^3/a^2$, *there exist integers* x *and* y *such that*

$$n < a x^2 + c y^2 < n + 2^{3/2} a^{1/2} c^{1/4} n^{1/4}.$$

Remark. Theorem 1 in this case says that for every integer $n \geqq a$ there exist integers x and y such that

$$n < a x^2 + c y^2 < n + 2^{3/2} a^{1/2} c^{1/4} n^{1/4} + a.$$

Proof of Theorem 2. Let n be an integer $\geqq c^3/a^2$, and set

$$E = 2 c^{1/2} n^{1/2} - 2^{1/2} a^{1/2} c^{1/4} n^{1/4} + \frac{a}{4}.$$

First we show that

(1) $\qquad E > 0.$

We have

$$\begin{aligned} E &= 2^{1/2} c^{1/4} n^{1/4} (2^{1/2} c^{1/4} n^{1/4} - a^{1/2}) + \frac{a}{4} \\ &> \frac{2^{1/2} c}{a^{1/2}} \left(\frac{2^{1/2} c}{a^{1/2}} - a^{1/2} \right) \qquad (\text{as } n \geqq c^3/a^2) \\ &\geqq 2^{1/2} a (2^{1/2} - 1) \qquad (\text{as } c \geqq a) \\ &> 0, \end{aligned}$$

proving (1).

Next we show that

(2) $$n - E > (n^{1/2} - c^{1/2})^2.$$

We have

$$\begin{aligned} 2^{1/2} a^{1/2} c^{1/4} n^{1/4} &\geqq 2^{1/2} c \qquad (\text{as } n \geqq c^3/a^2) \\ &> \frac{5}{4} c \\ &\geqq \frac{a}{4} + c \qquad (\text{as } c \geqq a) \end{aligned}$$

and so

$$n - E = (n^{1/2} - c^{1/2})^2 + \left(2^{1/2} a^{1/2} c^{1/4} n^{1/4} - \left(\frac{a}{4} + c \right) \right) > (n^{1/2} - c^{1/2})^2,$$

proving (2).

Thus we can define a real number α by

(3) $$\alpha = \frac{n^{1/2} - (n - E)^{1/2}}{c^{1/2}}.$$

From (1), (2) and (3), we see that

(4) $$0 < \alpha < 1.$$

We also note that

(5) $$\alpha^2 = \frac{2 n^{1/2}}{c^{1/2}} \alpha - \left(\frac{2 n^{1/2}}{c^{1/2}} - \frac{2^{1/2} a^{1/2} n^{1/4}}{c^{3/4}} + \frac{a}{4 c} \right).$$

Further we have

$$\begin{aligned} \alpha &= \frac{E}{c^{1/2} (n^{1/2} + (n - E)^{1/2})} \\ &> \frac{E}{2 c^{1/2} n^{1/2}} \\ &= 1 - \frac{a^{1/2}}{2^{1/2} c^{1/4} n^{1/4}} + \frac{a}{8 c^{1/2} n^{1/2}} \\ &> \frac{7}{8} - \frac{a^{1/2}}{2^{1/2} c^{1/4} n^{1/4}} \\ &\geqq \frac{7}{8} - \frac{a^{1/2} n^{1/4}}{2^{1/2} c^{3/4}} \qquad \left(\text{as } n \geqq \frac{c^3}{a^2} \geqq c \right) \end{aligned}$$

so that

$$(6)\qquad -2c\alpha+\frac{7c}{4}<2^{1/2}a^{1/2}c^{1/4}n^{1/4}.$$

We set

$$\delta=\left\{\left(\frac{n}{c}\right)^{1/2}\right\}\quad(\text{so that } 0\leqq\delta<1)$$

and consider two cases.

Case (i): $\alpha\leqq\delta$. We choose x and y to be the integers

$$x=1,\quad y=\left[\left(\frac{n}{c}\right)^{1/2}\right]+1=\left(\frac{n}{c}\right)^{1/2}+1-\delta,$$

so that

$$ax^2+cy^2=n+c(1-\delta)^2+2c^{1/2}n^{1/2}(1-\delta)+a.$$

Clearly we have $n<ax^2+cy^2$. Further we have

$$\begin{aligned}ax^2+cy^2&\leqq n+c(1-\alpha)^2+2c^{1/2}n^{1/2}(1-\alpha)+a &&(\text{as } \alpha\leqq\delta<1)\\ &=n+c\alpha^2-(2c+2c^{1/2}n^{1/2})\alpha+(c+2c^{1/2}n^{1/2}+a)\\ &=n+2^{1/2}a^{1/2}c^{1/4}n^{1/4}-2c\alpha+c+\frac{3a}{4} &&(\text{by (5)})\\ &\leqq n+2^{1/2}a^{1/2}c^{1/4}n^{1/4}-2c\alpha+\frac{7c}{4} &&(\text{as } c\geqq a)\\ &<n+2^{3/2}a^{1/2}c^{1/4}n^{1/4} &&(\text{by (6)}).\end{aligned}$$

Case (ii): $\delta<\alpha$. We choose x and y to be the integers

$$x=\left[\left(\frac{n-cy^2}{a}\right)^{1/2}\right]+1,\quad y=\left[\left(\frac{n}{c}\right)^{1/2}\right]=\left(\frac{n}{c}\right)^{1/2}-\delta.$$

Set

$$\varepsilon=\left\{\left(\frac{n-cy^2}{a}\right)^{1/2}\right\},$$

so that

$$0\leqq\varepsilon<1,\quad x=\left(\frac{n-cy^2}{a}\right)^{1/2}+(1-\varepsilon),$$

and

$$ax^2+cy^2=n+2(1-\varepsilon)a^{1/2}c^{1/4}(2n^{1/2}\delta-c^{1/2}\delta^2)^{1/2}+a(1-\varepsilon)^2.$$

Clearly $a x^2 + c y^2 > n$. Further we have

$$\begin{aligned} a x^2 + c y^2 &\leqq n + 2 a^{1/2} c^{1/4} (2 n^{1/2} \delta - c^{1/2} \delta^2)^{1/2} + a \\ &< n + 2 a^{1/2} c^{1/4} (2 n^{1/2} \alpha - c^{1/2} \alpha^2)^{1/2} + a \quad (\text{as } \delta < \alpha < 1 \text{ and } n \geqq c) \\ &= n + 2 a^{1/2} c^{1/4} \left(2^{1/2} n^{1/4} - \frac{a^{1/2}}{2 c^{1/4}} \right) + a \quad (\text{by (5)}) \\ &= n + 2^{3/2} a^{1/2} c^{1/4} n^{1/4}. \end{aligned}$$

This completes the proof of Theorem 2. □

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