

**A DETERMINISTIC ALGORITHM FOR SOLVING
 $n = fu^2 + gv^2$ IN COPRIME INTEGERS u AND v**

KENNETH HARDY, JOSEPH B. MUSKAT, AND KENNETH S. WILLIAMS

ABSTRACT. We give a deterministic algorithm for finding all primitive representations of a natural number n in the form $fu^2 + gv^2$, where f and g are given positive coprime integers, and $n \geq f + g + 1$, $(n, fg) = 1$. The running time of this algorithm is at most

$$O(n^{1/4}(\log n)^3(\log \log n)(\log \log \log n)),$$

uniformly in f and g .

1. INTRODUCTION

Throughout this paper, f and g denote integers such that

$$(1.1) \quad f \geq 1, \quad g \geq 1, \quad (f, g) = 1,$$

and n denotes an integer such that

$$(1.2) \quad n \geq f + g + 1, \quad (n, fg) = 1.$$

We are interested in the problem of determining all positive integers u and v (if any) such that

$$(1.3) \quad n = fu^2 + gv^2, \quad (u, v) = 1.$$

If (u, v) is a solution of (1.3) in positive integers, then

$$(1.4) \quad (u, n) = (v, n) = 1,$$

and

$$(1.5) \quad u \neq v.$$

In view of (1.4), we see that $v^{-1} \pmod{n}$ exists, and so we can define an integer y by $y \equiv uv^{-1} \pmod{n}$, $0 \leq y < n$. Clearly, $(n, y) = 1$, and y is a solution of $fy^2 \equiv -g \pmod{n}$. In particular, we have $y \neq 0$, and $y \neq n/2$ if n is even. Replacing y by $n - y$, if necessary, we obtain a solution y of $fy^2 \equiv -g \pmod{n}$ satisfying $y \equiv \pm uv^{-1} \pmod{n}$ and $0 < y < n/2$.

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Conversely, suppose y is a solution of

$$(1.6) \quad fy^2 \equiv -g \pmod{n}, \quad 0 < y < n/2.$$

Note that if (1.6) is insolvable, then so is (1.3). We define the (possibly empty) set $U(f, g, n, y)$ to be the set of pairs of integers (u, v) satisfying:

$$(1.7) \quad \begin{cases} n = fu^2 + gv^2, & (u, v) = 1, & uv^{-1} \equiv \pm y \pmod{n}, \\ \text{with } \begin{cases} u \geq 1, v \geq 1 & \text{if } fg \geq 2, \\ u > v \geq 1 & \text{if } fg = 1. \end{cases} \end{cases}$$

It is easily shown along the lines of the proof given in [18, pp. 332–335] that either $U(f, g, n, y)$ is empty or contains exactly one pair of integers. The main result of this paper is the following theorem which is proved in §2.

Theorem 1. *Let y be an integer satisfying (1.6) for which $U(f, g, n, y)$ is nonempty. Let*

$$(1.8) \quad r_0 (= y) > r_1 > \cdots > r_{s-1} (= 1) > r_s (= 0)$$

be the remainders obtained by applying the Euclidean algorithm to y and n . Let r_k ($0 \leq k \leq s$) be the first remainder $< \sqrt{n/f}$. Then we have

$$(1.9) \quad U(f, g, n, y) = \{(u, v)\},$$

where

$$(1.10) \quad u = r_k, \quad v = \sqrt{(n - fr_k^2)/g}.$$

Theorem 1 enables us to give a simple algorithm for finding all the solutions (if any) of (1.3) in positive integers u and v as follows:

Algorithm. First determine all the solutions y of $fy^2 \equiv -g \pmod{n}$, $0 < y < n/2$. For each solution y , apply the Euclidean algorithm to y and n , and let $r = r(f, g, n, y)$ denote the first remainder $< \sqrt{n/f}$. Then all solutions (u, v) of (1.3) (with $u > v$ if $fg = 1$) lie among the pairs $(r, \sqrt{(n - fr^2)/g})$.

A deterministic version of this algorithm is described and analyzed in §4, see Theorem 3.

Theorem 2 below, which is proved in §3, gives an alternative expression for v to that given in (1.10). We remark that Theorem 2 is not needed in the algorithm.

Theorem 2. *With the notation of Theorem 1, we have*

$$(1.11) \quad v = \begin{cases} 1 & \text{if } k = 0, \\ r_{k+1} & \text{if } k \geq 1, fg = 1, \\ (r_{k-1} - cr_k)/g & \text{if } k \geq 1, fg \geq 2, \end{cases}$$

and c is the unique integer satisfying

$$(1.12) \quad r_k c \equiv r_{k-1} + (-1)^k f y r_k \pmod{n}, \quad 0 \leq c < n.$$

We remark that Brillhart's modification [3] of the Hermite-Serret algorithm [8, 15] for solving $p = u^2 + v^2$, where $p \equiv 1 \pmod{4}$ is prime, is a special case of Theorem 2. Our algorithm also contains those of Cornacchia [6] and Wilker [19] as special cases. Our proof is different from that of Brillhart in that the palindromic nature of the continued fraction used in [3] does not always hold.

2. THE INTEGERS c_i AND d_i AND THE PROOF OF THEOREM 1

Let y be an integer satisfying (1.6) for which

$$(2.1) \quad U(f, g, n, y) \neq \emptyset.$$

By the remarks following (1.7), there is a *unique* pair of integers (u, v) such that

$$(2.2) \quad \begin{cases} n = fu^2 + gv^2, & (u, v) = 1, \quad uv^{-1} \equiv \pm y \pmod{n}, \\ \text{with } \begin{cases} u \geq 1, v \geq 1 & \text{if } fg \geq 2, \\ u > v \geq 1 & \text{if } fg = 1. \end{cases} \end{cases}$$

We define ε ($= \pm 1$) to be the *unique* integer satisfying

$$(2.3) \quad y \equiv \varepsilon uv^{-1} \pmod{n}.$$

Applying the Euclidean algorithm to y and n , we obtain

$$(2.4) \quad \begin{cases} y = q_0n + r_0, \\ n = q_1r_0 + r_1, \\ r_{i-2} = q_i r_{i-1} + r_i \quad (i = 2, \dots, s), \end{cases}$$

where

$$(2.5) \quad s \geq 1,$$

$$(2.6) \quad r_0 (= y) > r_1 > r_2 > \dots > r_{s-1} (= 1) > r_s (= 0),$$

and

$$(2.7) \quad \begin{aligned} q_0 = [y/n] = 0, \quad q_1 = [n/r_0] = [n/y] \geq 2, \\ q_i = [r_{i-2}/r_{i-1}] \geq 1 \quad (i = 2, \dots, s). \end{aligned}$$

The continued fraction for y/n is

$$(2.8) \quad \frac{y}{n} = [q_0, q_1, q_2, \dots, q_s].$$

The i th convergent to y/n is

$$(2.9) \quad \frac{A_i}{B_i} = [q_0, q_1, q_2, \dots, q_i] \quad (i = 0, 1, \dots, s),$$

so that, in particular, we have

$$(2.10) \quad \begin{aligned} A_0 = 0, \quad B_0 = 1, \\ A_1 = 1, \quad B_1 = q_1, \\ A_2 = q_2, \quad B_2 = q_1q_2 + 1, \\ \vdots \quad \quad \quad \vdots \\ A_s = y, \quad B_s = n. \end{aligned}$$

Moreover, we have

$$(2.11) \quad \begin{aligned} A_i &= q_i A_{i-1} + A_{i-2} & (i = 2, \dots, s), \\ B_i &= q_i B_{i-1} + B_{i-2} & (i = 2, \dots, s). \end{aligned}$$

From (2.4) and (2.11), we obtain, for $i = 1, \dots, s-1$,

$$(B_{i+1} - B_{i-1})r_i = q_{i+1}B_i r_i = (r_{i-1} - r_{i+1})B_i,$$

so that

$$r_i B_{i+1} + r_{i+1} B_i = r_{i-1} B_i + r_i B_{i-1} \quad (i = 1, \dots, s-1),$$

and so, for $i = 1, \dots, s-1$, we have

$$r_i B_{i+1} + r_{i+1} B_i = r_0 B_1 + r_1 B_0 = r_0 q_1 + r_1 = n,$$

and thus

$$(2.12) \quad r_i B_{i+1} + r_{i+1} B_i = n \quad (i = 0, 1, \dots, s-1).$$

An easy induction argument on i , using (2.4), (2.6), and (2.11), shows that

$$(2.13) \quad r_i = (-1)^i (B_i y - A_i n) \quad (i = 0, 1, \dots, s),$$

so that

$$(2.14) \quad r_i \equiv (-1)^i B_i y \pmod{n} \quad (i = 0, 1, \dots, s).$$

From (2.2), (2.3), and (2.14), we see that for $i = 0, 1, \dots, s$,

$$(2.15) \quad \begin{cases} fr_i u + \varepsilon(-1)^i g B_i v \equiv 0 \pmod{n}, \\ r_i v - \varepsilon(-1)^i B_i u \equiv 0 \pmod{n}. \end{cases}$$

Hence, we may define integers c_i and d_i ($i = 0, 1, \dots, s$) by

$$(2.16) \quad \begin{cases} c_i = (fr_i u + \varepsilon(-1)^i g B_i v)/n, \\ d_i = (r_i v - \varepsilon(-1)^i B_i u)/n. \end{cases}$$

Using (2.2), (2.4), (2.11), and (2.16), it is easy to show that

$$(2.17) \quad \begin{cases} c_i = -q_i c_{i-1} + c_{i-2} & (i = 2, 3, \dots, s), \\ d_i = -q_i d_{i-1} + d_{i-2} & (i = 2, 3, \dots, s), \end{cases}$$

$$(2.18) \quad c_i^2 + fg d_i^2 = (fr_i^2 + g B_i^2)/n \quad (i = 0, 1, \dots, s),$$

and

$$(2.19) \quad c_i d_{i+1} - c_{i+1} d_i = (-1)^i \varepsilon \quad (i = 0, 1, \dots, s-1).$$

We note that

$$(2.20) \quad c_0 = (fyu + \varepsilon gv)/n, \quad d_0 = (yv - \varepsilon u)/n,$$

$$(2.21) \quad c_1 = fu - q_1 c_0, \quad d_1 = v - q_1 d_0,$$

$$(2.22) \quad c_s = \varepsilon(-1)^s gv, \quad d_s = \varepsilon(-1)^{s+1} u.$$

We emphasize that c_s and d_s are nonzero.

Lemma 1. Suppose that $c_i = 0$ for some integer i with $0 \leq i \leq s - 1$. Set

$$a = \begin{cases} 0, & i \text{ even,} \\ 1, & i \text{ odd,} \end{cases} \quad b = \begin{cases} 0, & s - i \text{ even,} \\ 1, & s - i \text{ odd.} \end{cases}$$

Then we have $\varepsilon = (-1)^{i+1}$ and

- (a) $c_a > \dots > c_{i-2} > c_i = 0 > c_{i+2} > \dots > c_{s-b}$,
- (b) $c_{1-a} > \dots > c_{i-3} > c_{i-1} = 1 = c_{i+1} < c_{i+3} < \dots < c_{s-(1-b)}$,
- (c) $d_{1-a} > \dots > d_{i-1} \geq 0 \geq d_{i+1} > d_{i+3} > \dots > d_{s-(1-b)}$, where at most one of the equality signs holds, and
- (d) $d_a > \dots > d_{i-2} \geq d_i = 1 \leq d_{i+2} < \dots < d_{s-b}$.

Proof. As $c_i = 0$, appealing to (2.16), we obtain $fr_iu = \varepsilon(-1)^{i+1}gB_iv$. Now $r_i \geq 1$, as $0 \leq i \leq s - 1$, and so, as f, g, u, v, B_i are all positive, we see that $\varepsilon = (-1)^{i+1}$.

For $0 \leq i \leq s - 1$, from (2.19), we obtain $c_{i+1}d_i = (-1)^{i+1}\varepsilon = 1$. As c_{i+1} and d_i are both integers, we must have $c_{i+1} = d_i = \pm 1$. But, from (2.16), we obtain $d_i = (r_iv + B_iu)/n > 0$, so we must have

$$c_{i+1} = d_i = 1 \quad (0 \leq i \leq s - 1).$$

For $1 \leq i \leq s - 1$, from (2.19), we obtain (as $c_i = 0, d_i = 1$)

$$c_{i-1} = 1 \quad (1 \leq i \leq s - 1).$$

For $0 \leq i \leq s - 2$, from (2.16), we have

$$c_i - c_{i+2} = (fu(r_i - r_{i+2}) + gv(B_{i+2} - B_i))/n > 0,$$

so that $c_{i+2} < 0$ ($0 \leq i \leq s - 2$).

For $2 \leq i \leq s - 1$, from (2.16), we obtain

$$c_{i-2} - c_i = (fu(r_{i-2} - r_i) + gv(B_i - B_{i-2}))/n > 0,$$

so that $c_{i-2} > 0$ ($2 \leq i \leq s - 1$).

For $2 \leq i \leq s - 1$, from (2.16), we obtain

$$d_{i-2} = (r_{i-2}v + B_{i-2}u)/n > 0,$$

so that $d_{i-2} \geq 1$ ($2 \leq i \leq s - 1$). Further, for $2 \leq i \leq s - 1$, appealing to (2.19), we obtain

$$c_{i-2}d_{i-1} - c_{i-1}d_{i-2} = \varepsilon(-1)^{i-2} = -1,$$

so that $c_{i-2}d_{i-1} = d_{i-2} - 1 \geq 0$, and thus $d_{i-1} \geq 0$ ($2 \leq i \leq s - 1$). This inequality is also true for $i = 1$ as we now show. When $i = 1$ we have $\varepsilon = 1, c_0 = 1$, and so, by (2.20), $n = nc_0 = fyu + gv$ and

$$\begin{aligned} d_0 &= \frac{yv - u}{n} = \frac{fuyv - fu^2}{fun} = \frac{fuyv - (n - gv^2)}{fun} \\ &= \frac{v(fuy + gv) - n}{fun} = \frac{v - 1}{fu} \geq 0. \end{aligned}$$

Hence we have $d_{i-1} \geq 0$ ($1 \leq i \leq s - 1$).

For $0 \leq i \leq s - 2$, from (2.16), we obtain $d_{i+2} = (r_{i+2}v + B_{i+2}u)/n > 0$, so that $d_{i+2} \geq 1$ ($0 \leq i \leq s - 2$). Further, for $0 \leq i \leq s - 2$, from (2.19), we have $c_{i+1}d_{i+2} - c_{i+2}d_{i+1} = 1$, so that $c_{i+2}d_{i+1} = d_{i+2} - 1 \geq 0$, and thus $d_{i+1} \leq 0$ ($0 \leq i \leq s - 2$). But the last inequality also holds for $i = s - 1$, as $d_{i+1} = d_s = \varepsilon(-1)^{s+1}u = -u < 0$. Hence we have $d_{i+1} \leq 0$ ($0 \leq i \leq s - 1$).

For $a \leq t \leq s - b - 2$ and $t \equiv i \pmod{2}$, we have, by (2.16),

$$c_t - c_{t+2} = (fu(r_t - r_{t+2}) + gv(B_{t+2} - B_t))/n > 0,$$

so that $c_t > c_{t+2}$ ($a \leq t \leq s - b - 2$, $t \equiv i \pmod{2}$). This completes the proof of (a).

For $1 - a \leq t \leq i - 3$ and $t \equiv i + 1 \pmod{2}$, we have, by (2.17) and (a),

$$c_t - c_{t+2} = q_{t+2}c_{t+1} \geq q_{t+2}c_{i-2} > 0,$$

so that $c_t > c_{t+2}$ ($1 - a \leq t \leq i - 3$, $t \equiv i + 1 \pmod{2}$). For $i + 1 \leq t \leq s - 3 + b$ and $t \equiv i + 1 \pmod{2}$, we have, by (2.17) and (a),

$$c_t - c_{t+2} = q_{t+2}c_{t+1} \leq q_{t+2}c_{i+2} < 0,$$

so that $c_t < c_{t+2}$ ($i + 1 \leq t \leq s - 3 + b$, $t \equiv i + 1 \pmod{2}$). This completes the proof of (b).

For $1 - a \leq t \leq s - 3 + b$ and $t \equiv i + 1 \pmod{2}$, we have, by (2.16),

$$d_t - d_{t+2} = ((r_t - r_{t+2})v + (B_{t+2} - B_t)u)/n > 0,$$

so that $d_t > d_{t+2}$ ($1 - a \leq t \leq s - 3 + b$, $t \equiv i + 1 \pmod{2}$). This completes the proof of (c).

For $a \leq t \leq i - 4$ and $t \equiv i \pmod{2}$, we have, by (2.17) and (c),

$$d_t - d_{t+2} = q_{t+2}d_{t+1} \geq q_{t+2}d_{i-3} > q_{t+2}d_{i-1} \geq 0,$$

so that $d_t > d_{t+2}$ ($a \leq t \leq i - 4$, $t \equiv i \pmod{2}$).

For $i + 2 \leq t \leq s - 2 - b$ and $t \equiv i \pmod{2}$, we have, by (2.17) and (c),

$$d_t - d_{t+2} = q_{t+2}d_{t+1} \leq q_{t+2}d_{i+3} < q_{t+2}d_{i+1} \leq 0,$$

so that $d_t < d_{t+2}$ ($i + 2 \leq t \leq s - 2 - b$, $t \equiv i \pmod{2}$). This completes the proof of (d). The proof of Lemma 1 is now complete. \square

Lemma 2. Suppose that $d_i = 0$ for some integer i with $0 \leq i \leq s - 1$. Set

$$a = \begin{cases} 0, & i \text{ even,} \\ 1, & i \text{ odd,} \end{cases} \quad b = \begin{cases} 0, & s - i \text{ even,} \\ 1, & s - i \text{ odd.} \end{cases}$$

Then we have $\varepsilon = (-1)^i$ and

- (a) $c_{1-a} > \dots > c_{i-1} \geq 0 \geq c_{i+1} > \dots > c_{s-(1-b)}$, where at most one of the equality signs holds,
- (b) $c_a > \dots > c_{i-4} > c_{i-2} \geq c_i = 1 \leq c_{i+2} < c_{i+4} < \dots < c_{s-b}$,
- (c) $d_a > \dots > d_{i-2} > d_i = 0 > d_{i+2} > \dots > d_{s-b}$, and
- (d) $d_{1-a} > \dots > d_{i-3} > d_{i-1} = 1 = d_{i+1} < d_{i+3} < \dots < d_{s-(1-b)}$.

Proof. The proof of Lemma 2 is similar to that of Lemma 1 and will be omitted. \square

Lemma 3. (a) If $c_p = c_q = 0$ with $0 \leq p \leq q \leq s - 1$, then $p = q$.

(b) If $d_p = d_q = 0$ with $0 \leq p \leq q \leq s - 1$, then $p = q$.

(c) If $c_p = d_q = 0$ with $0 \leq p \leq s - 1$, $0 \leq q \leq s - 1$, then either $p = q + 1$ or $p = q - 1$.

Proof. (a) Immediate from Lemma 1(a), (b).

(b) Immediate from Lemma 2(c), (d).

(c) Immediate from Lemma 1 (or from Lemma 2). \square

We now define the nonnegative integers k and j which are central to the proofs of Theorems 1 and 2. We let r_k ($0 \leq k \leq s$) be the largest remainder which is less than $\sqrt{n/f}$, and B_j ($0 \leq j \leq s$) the largest denominator of the convergents to y/n which is less than $\sqrt{n/g}$. Clearly, $r_{s-1} = 1 < \sqrt{n/f}$, showing that $0 \leq k \leq s - 1$. Also we have $\sqrt{n/g} \leq \sqrt{n} < n = B_s$, so that $0 \leq j \leq s - 1$.

If $k = 0$, then $y = r_0 < \sqrt{n/f}$, $fy^2 + g \equiv 0 \pmod{n}$, $fy^2 + g < 2n$, so that $n = fy^2 + g1^2$, showing that $(y, 1) \in U(f, g, n, y)$. Hence we have $(y, 1) = (u, v)$, and so in the case $k = 0$ we have

$$(2.23) \quad u = y = r_0, \quad v = 1 = B_0,$$

as asserted in Theorems 1 and 2. When $k = 0$ we also show that

$$j = \begin{cases} 0 & \text{if } fg \geq 2, \\ 1 & \text{if } fg = 1, \end{cases}$$

as follows. From $n = fy^2 + g1^2$ and $n > f + g + 1$, we obtain $y^2 \geq 1 + 1/f > 1$, so that (as $0 < y < n/2$) we have $y \geq 2$.

We first treat the case $fg \geq 2$. We suppose that $j \geq 1$ and obtain a contradiction. We have

$$\begin{aligned} fy \leq fy + \left[\frac{g}{y} \right] &= \left[fy + \frac{g}{y} \right] = \left[\frac{fy^2 + g}{y} \right] = \left[\frac{n}{y} \right] = q_1 \\ &= B_1 \leq B_j < \sqrt{\frac{n}{g}} = \sqrt{\frac{fy^2 + g}{g}} = \sqrt{1 + \frac{fy^2}{g}}, \end{aligned}$$

so that

$$f^2 y^2 < 1 + \frac{fy^2}{g}.$$

If $f = 1$, we have $g \geq 2$, and the inequality becomes

$$y^2 < 1 + \frac{y^2}{g} \leq 1 + \frac{y^2}{2},$$

which is impossible, as $y \geq 2$. On the other hand, if $f \geq 2$, the inequality gives

$$2y^2 \leq f(f - 1)y^2 < 1 - fy^2(g - 1)/g < 1,$$

which is again impossible. Hence we must have $j = 0$ as claimed.

Next we treat the case $fg = 1$, that is, $f = g = 1$. In this case we have $n = fy^2 + g1^2 = y^2 + 1$, where $y \geq 2$, and the Euclidean algorithm applied to y and n just consists of three lines, namely

$$\begin{cases} y = 0n + y, \\ n = y1 + 1, \\ y = y1 + 0, \end{cases}$$

so that $s = 2$, $r_0 = y$, $r_1 = 1$, $r_2 = 0$, $q_0 = 0$, and $q_1 = q_2 = y$. Thus we have

$$B_1 = q_1 = y < \sqrt{n} < n = B_2,$$

and $j = 1$ as claimed.

This completes the treatment of the case $k = 0$, and so from here until the end of §3, we may assume $k \geq 1$. Thus we have

$$(2.24) \quad r_k < \sqrt{n/f} \leq r_{k-1} \quad (1 \leq k \leq s-1),$$

and

$$(2.25) \quad B_j < \sqrt{n/g} \leq B_{j+1} \quad (0 \leq j \leq s-1).$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. From (2.12) and (2.24), we obtain

$$\sqrt{n/f}B_k \leq r_{k-1}B_k \leq r_{k-1}B_k + r_kB_{k-1} = n,$$

so that

$$(2.26) \quad B_k \leq \sqrt{fn}.$$

From (2.12) and (2.25), we obtain

$$\sqrt{n/gr_j} \leq r_jB_{j+1} \leq r_jB_{j+1} + r_{j+1}B_j = n,$$

so that

$$(2.27) \quad r_j \leq \sqrt{gn}.$$

Then, by (2.18), (2.24), and (2.26), we have

$$(2.28) \quad c_k^2 + fg d_k^2 = (fr_k^2 + gB_k^2)/n < 1 + fg,$$

and, by (2.18), (2.25), and (2.27), we have

$$(2.29) \quad c_j^2 + fg d_j^2 = (fr_j^2 + gB_j^2)/n < fg + 1.$$

Hence, from (2.28), we deduce

$$(2.30) \quad d_k = 0 \quad \text{or} \quad d_k = \pm 1, \quad c_k = 0,$$

and, from (2.29), we deduce

$$(2.31) \quad d_j = 0 \quad \text{or} \quad d_j = \pm 1, \quad c_j = 0.$$

We first show that neither of the possibilities

- (a) $d_k = \pm 1, c_k = 0, d_j = 0,$
- (b) $d_k = \pm 1, c_k = 0, d_j = \pm 1, c_j = 0,$

can occur.

(a) $d_k = \pm 1, c_k = 0, d_j = 0.$ By Lemma 3(c) we have $j = k + 1$ or $j = k - 1.$ First, suppose that $j = k + 1.$ By Lemma 1(b), (d), we have

$$c_k = 0, \quad d_k = 1, \quad c_{k+1} = 1, \quad d_{k+1} = 0, \quad \varepsilon = (-1)^{k+1}.$$

Appealing to (2.16), we obtain

$$fur_k - gvB_k = 0, \quad vr_k + uB_k = n,$$

and

$$fur_{k+1} + gvB_{k+1} = n, \quad vr_{k+1} - uB_{k+1} = 0.$$

Solving these linear equations for r_k, B_k and $r_{k+1}, B_{k+1},$ we obtain

$$r_k = gv, \quad B_k = fu, \quad r_{k+1} = u, \quad B_{k+1} = v.$$

As $r_{k+1} < r_k < \sqrt{n/f}$ and $B_k < B_{k+1} < \sqrt{n/g},$ we deduce that

$$u < gv < \sqrt{n/f}, \quad fu < v < \sqrt{n/g}.$$

Further, as $u > v$ for $fg = 1,$ we see that $fg \geq 2.$ Then we have

$$1 + fg < (fg)^2 < \frac{fn}{v^2} = \frac{f^2u^2}{v^2} + fg < 1 + fg,$$

which is impossible.

Next, suppose that $j = k - 1.$ By Lemma 1(b), (d), we have

$$c_k = 0, \quad d_k = 1, \quad c_{k-1} = 1, \quad d_{k-1} = 0, \quad \varepsilon = (-1)^{k+1}.$$

Appealing to (2.16), we obtain

$$fur_k - gvB_k = 0, \quad vr_k + uB_k = n,$$

and

$$fur_{k-1} + gvB_{k-1} = n, \quad vr_{k-1} - uB_{k-1} = 0.$$

Solving these linear equations for r_k, B_k and $r_{k-1}, B_{k-1},$ we obtain

$$r_k = gv, \quad B_k = fu, \quad r_{k-1} = u, \quad B_{k-1} = v.$$

Appealing to (2.24), we obtain $\sqrt{n/f} \leq u,$ and so

$$n = fu^2 + gv^2 > fu^2 \geq n,$$

which is impossible. Thus case (a) cannot occur.

(b) $d_k = \pm 1, c_k = 0, d_j = \pm 1, c_j = 0.$ By Lemma 3(a) we have $j = k,$ and by Lemma 1(d) we have $c_k = 0, d_k = 1,$ and $\varepsilon = (-1)^{k+1}.$ Appealing to (2.16), we obtain

$$fur_k - gvB_k = 0, \quad vr_k + uB_k = n.$$

Solving these linear equations for r_k and B_k , we obtain $r_k = gv$, $B_k = fu$. Then, from (2.24) and (2.25), we deduce

$$gv < \sqrt{n/f}, \quad fu < \sqrt{n/g}.$$

If $fg \geq 2$, then we have

$$n = fu^2 + gv^2 < \frac{n}{fg} + \frac{n}{fg} = \frac{2n}{fg} \leq n,$$

which is impossible. Thus we must have $fg = 1$, so that

$$f = g = 1, \quad r_k = v, \quad B_k = u.$$

Then, as $c_{k-1} = 1$ by Lemma 1(b), we obtain from (2.12) (with $i = k - 1$) $n = ur_{k-1} + vB_{k-1}$. Thus, as $(u, v) = 1$, there is an integer t such that

$$r_{k-1} = u + vt, \quad B_{k-1} = v - ut.$$

As $0 < B_{k-1} \leq B_k$ and $u > v > 0$, we have $-1 < v/u - 1 \leq t < v/u < 1$, so that $t = 0$, and thus $r_{k-1} = u$, $B_{k-1} = v$. Then, from (2.24) and (2.25), we obtain the contradiction $\sqrt{n} \leq u$, $u < \sqrt{n}$. Thus case (b) cannot occur.

Hence, from (2.30) and (2.31), we see that we must have $d_k = 0$. By Lemma 2(b) we have $\varepsilon = (-1)^k$ and $c_k = 1$. Appealing to (2.16) (with $i = k$), we obtain

$$fur_k + gvB_k = n, \quad vr_k - uB_k = 0.$$

Solving these linear equations for r_k and B_k , we obtain $r_k = u$, $B_k = v$. This completes the proof of Theorem 1. \square

We finish this section by showing that

$$(2.32) \quad j = \begin{cases} k & \text{if } fg \geq 2, \\ k + 1 & \text{if } fg = 1. \end{cases}$$

Equation (2.32) has already been proved when $k = 0$, so we may assume that $k \geq 1$. We have shown that

$$(2.33) \quad c_k = 1, \quad d_k = 0, \quad \varepsilon = (-1)^k, \quad r_k = u, \quad B_k = v.$$

Hence we have $B_k = v < \sqrt{n/g}$, so that

$$(2.34) \quad j \geq k.$$

From (2.31) we see that either

$$(2.35) \quad d_j = 0$$

or

$$(2.36) \quad c_j = 0, \quad d_j = \pm 1.$$

If (2.36) holds, we have $c_j = d_k = 0$, and so by Lemma 2(c) we have $j = k + 1$ or $j = k - 1$. The second possibility is excluded by (2.34). Thus, $j = k + 1$

and $c_{k+1} = 0$, $d_{k+1} = \pm 1$. By Lemma 1(d) we deduce $d_{k+1} = 1$. Appealing to (2.16), we obtain

$$fur_{k+1} - gvB_{k+1} = 0, \quad vr_{k+1} + uB_{k+1} = n.$$

Solving for r_{k+1} , B_{k+1} , we deduce $r_{k+1} = gv$, $B_{k+1} = fu$. Since $r_{k+1} < r_k < \sqrt{n/f}$ and $B_k < B_{k+1} < \sqrt{n/g}$, we have $gv < u < \sqrt{n/f}$ and $v < fu < \sqrt{n/g}$, so that

$$n = fu^2 + gv^2 < f \left(\frac{1}{f} \sqrt{\frac{n}{g}} \right)^2 + g \left(\frac{1}{g} \sqrt{\frac{n}{f}} \right)^2 = \frac{n}{fg} + \frac{n}{fg} = \frac{2n}{fg}.$$

But this is a contradiction when $fg \geq 2$. Hence (2.35) holds when $fg \geq 2$ and so, by Lemma 3(b), we have $j = k$ as asserted.

Finally, we treat the case $fg = 1$, that is, $f = g = 1$. If (2.35) holds, we have $d_k = d_j = 0$, and so, by Lemma 3(b), we have $j = k$. By Lemma 2(a), (d), we obtain $0 \geq c_{k+1}$, $d_{k+1} = 1$. By (2.16) we have

$$ur_{k+1} - vB_{k+1} = c_{k+1}n, \quad vr_{k+1} + uB_{k+1} = n.$$

Solving for r_{k+1} , B_{k+1} , we deduce

$$r_{k+1} = v + c_{k+1}u, \quad B_{k+1} = u - c_{k+1}v.$$

Since $0 \leq r_{k+1} < r_k$ and $B_k < \sqrt{n} \leq B_{k+1}$, we have

$$(2.37) \quad 0 \leq v + c_{k+1}u < u$$

and

$$(2.38) \quad v < \sqrt{n} \leq u - c_{k+1}v.$$

But, by (1.7), we have $u > v$, so that by (2.37)

$$-1 < -v/u \leq c_{k+1} < 1 - v/u < 1,$$

and so $c_{k+1} = 0$. But then from (2.38) we have $u \geq \sqrt{n}$, contradicting $u < \sqrt{n}$. Hence, (2.36) must hold, and so $j = k + 1$ as before.

This completes the proof of (2.32). \square

3. PROOF OF THEOREM 2

In §2 we showed that $c_k = 1$, $d_k = 0$, $\varepsilon = (-1)^k$, $u = r_k$, and $v = B_k$. Thus, to prove Theorem 2, we must show that

$$B_k = \begin{cases} r_{k+1} & \text{if } k \geq 1, fg = 1, \\ (r_{k-1} - cr_k)/g & \text{if } k \geq 1, fg \geq 2. \end{cases}$$

We first suppose that $k \geq 1$ and $fg = 1$. From the analysis at the end of §2 we have $c_{k+1} = 0$, and $r_{k+1} = v$, $B_{k+1} = u$. Hence, $B_k = v = r_{k+1}$ as required.

Next we suppose that $k \geq 1$ and $fg \geq 2$. As $(u, n) = 1$, we have $(r_k, n) = 1$, and so the congruence

$$r_k c \equiv r_{k-1} + (-1)^k f y r_k \pmod{n}, \quad 0 \leq c < n,$$

has a *unique* solution c . From (2.16) (with $i = k - 1$), recalling that $\varepsilon = (-1)^k$ and noting that $d_{k-1} = 1$ by Lemma 2(d), we obtain

$$f u r_{k-1} - g v B_{k-1} = c_{k-1} n, \quad v r_{k-1} + u B_{k-1} = n,$$

so that $r_{k-1} = c_{k-1} u + g v$, $B_{k-1} = f u - c_{k-1} v$, and hence

$$v = (r_{k-1} - c_{k-1} r_k) / g.$$

Next, appealing to Lemma 2(a), we note that

$$0 \leq c_{k-1} \leq c_{k-1} v = f u - B_{k-1} < f u \leq f u^2 < n,$$

and modulo n we observe that

$$\begin{aligned} r_k c_{k-1} &\equiv u v^{-1} (v c_{k-1}) \equiv u v^{-1} (f r_k - B_{k-1}) \equiv \varepsilon y (f r_k - B_{k-1}) \\ &\equiv (-1)^k y f r_k - (-1)^k y B_{k-1} \equiv r_{k-1} + (-1)^k f y r_k. \end{aligned}$$

This shows that

$$r_k c_{k-1} \equiv r_{k-1} + (-1)^k f y r_k \pmod{n}, \quad 0 \leq c_{k-1} < n,$$

proving that $c_{k-1} = c$. The proof of Theorem 2 is now complete. \square

4. THE ALGORITHM

Step 1. Use the Adleman-Pomerance-Rumely primality test [1] on the integer n . This is a deterministic algorithm with a worst case running time of $\mathcal{O}((\log n)^{\alpha \log \log n})$, where $\alpha > 0$ is an absolute constant and the constant implied by the \mathcal{O} -symbol is also absolute. If n is composite, go to Step 2; else set $r = 1$, $n = p_1$, $a_1 = 1$ and go to Step 3.

Step 2. Factor n into primes. The fastest known, fully proven, deterministic factoring algorithm is the Pollard-Strassen method discussed by Pomerance in [11, §4]. This algorithm has a running time of

$$\mathcal{O}(n^{1/4} (\log n)^3 \log \log n \log \log \log n),$$

where the constant implied by the \mathcal{O} -symbol is absolute. This step in the algorithm is the dominant one. In practice, one would use one of the following methods: ρ method, $p - 1$ method, elliptic curve method, quadratic sieve, etc. (see [10, 11, 13]). Set

$$(4.1) \quad n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where a_1, a_2, \dots, a_r are r (≥ 1) positive integers and p_1, p_2, \dots, p_r are distinct primes with $p_1 = 2$ if n is even. Continue with Step 3.

Step 3. Calculate $f^{-1} \pmod{n}$ and determine $m \equiv f^{-1} g \pmod{n}$, $0 < m < n$. Note $(m, n) = 1$. The congruence (1.6) is equivalent to $y^2 \equiv -m \pmod{n}$.

(a) If n is odd go to (b). Else n is even, m is odd, $p_1 = 2$, $a_1 \geq 1$. If $a_1 = 2$ and $m \equiv 1 \pmod{4}$, or $a_1 \geq 3$ and $m \not\equiv 7 \pmod{8}$, the congruence

$$(4.2) \quad y_1^2 \equiv -m \pmod{2^{a_1}}$$

For $a_1 \geq 3$ the solutions of (4.2) are computed by means of the recurrence relation

$$(4.7) \quad \begin{cases} y_{1,3} = 1, \\ y_{1,k+1} \equiv (y_{1,k}^3 + (m+2)y_{1,k})/2 \pmod{2^k} \\ \qquad \qquad \qquad (k = 3, 4, \dots, a_1 - 1), \\ y_1 = y_{1,a_1}. \end{cases}$$

The four solutions of (4.2) are given by

$$(4.8) \quad y \equiv \pm y_1, \pm(y_1 + 2^{a_1-1}) \pmod{2^{a_1}},$$

and can be calculated in $\mathcal{O}(a_1)$ steps.

Step 4 has a running time of

$$\begin{aligned} \sum_{i=1}^r \mathcal{O}(p_i^{(4\sqrt{\varepsilon})^{-1}+\varepsilon}) \mathcal{O}(a_i \log p_i) &= \mathcal{O}\left(\sum_{i=1}^r a_i p_i^{(4\sqrt{\varepsilon})^{-1}+2\varepsilon}\right) \\ &= \mathcal{O}\left(n^{(4\sqrt{\varepsilon})^{-1}+2\varepsilon} \sum_{i=1}^r a_i\right) = \mathcal{O}\left(n^{(4\sqrt{\varepsilon})^{-1}+2\varepsilon} \log n\right) = \mathcal{O}(n^{(4\sqrt{\varepsilon})^{-1}+3\varepsilon}), \end{aligned}$$

where the constant implied by the \mathcal{O} -symbol depends only upon ε . Thus the time for Step 4 is $\mathcal{O}(n^{(4\sqrt{\varepsilon})^{-1}+\varepsilon})$ uniformly in f and g .

We remark that if Schoof's algorithm [14] is used for solving (4.4), Step 4 has a running time of $\mathcal{O}(\log^{10} n)$ but the implied constant depends (strongly) on m .

Step 5. The Chinese remainder theorem is used to find the 2^r solutions $y \pmod{n}$ of

$$(4.9) \quad y \equiv \pm y_i \pmod{p_i^{a_i}} \quad (i = 1, 2, \dots, r), \text{ if } n \equiv 1 \pmod{2};$$

the 2^{r-1} solutions $y \pmod{n}$ of

$$(4.10) \quad y \equiv \begin{cases} 1 \pmod{2}, \\ \pm y_i \pmod{p_i^{a_i}} \end{cases} \quad (i = 2, \dots, r), \text{ if } n \equiv 2 \pmod{4};$$

the 2^r solutions $y \pmod{n}$ of

$$(4.11) \quad y \equiv \begin{cases} \pm 1 \pmod{4}, \\ \pm y_i \pmod{p_i^{a_i}} \end{cases} \quad (i = 2, \dots, r), \text{ if } n \equiv 4 \pmod{8};$$

and the 2^{r+1} solutions $y \pmod{n}$ of

$$(4.12) \quad y \equiv \begin{cases} \pm y_1, \pm(y_1 + 2^{a_1-1}) \pmod{2^{a_1}}, \\ \pm y_i \pmod{p_i^{a_i}} \end{cases} \quad (i = 2, \dots, r), \text{ if } n \equiv 0 \pmod{8}.$$

The values y obtained are the solutions \pmod{n} of $y^2 \equiv -m \pmod{n}$. Step 5 can be accomplished in $\mathcal{O}(2^r \log^2 n) = \mathcal{O}(2^{\beta \log n / \log \log n})$ steps, where β is a positive absolute constant, and the constant implied by the \mathcal{O} -symbol is absolute.

Step 6. For each of the solutions y of $y^2 \equiv -m \pmod{n}$ with $0 < y < n/2$ found in Step 5, we apply the Euclidean algorithm to y and n , and determine the first remainder r less than $\sqrt{n/f}$. By Theorem 1 all the solutions (u, v) of $n = fu^2 + gv^2$, $(u, v) = 1$, in positive integers (with $u > v$ if $fg = 1$) lie among the pairs $(r, \sqrt{(n - fr^2)/g})$. They are easily found by checking whether $\sqrt{(n - fr^2)/g}$ is an integer. Step 6 takes $\mathcal{O}(2^r \log n) = \mathcal{O}(2^{\beta \log n / \log \log n})$ steps, where the implied constant is absolute.

We have thus proved the following theorem.

Theorem 3. *Let n, f, g be integers satisfying (1.1) and (1.2). Then there is a deterministic algorithm which gives all the solutions of $n = fu^2 + gv^2$ in positive coprime integers u and v , with a worst case running time of*

$$\mathcal{O}(n^{1/4}(\log n)^3(\log \log n)(\log \log \log n)),$$

where the constant implied by the \mathcal{O} -symbol is independent of f and g .

We remark that the worst case running time for a rigorous random version of our algorithm is

$$\mathcal{O}(2^{\beta \log n / \log \log n}) \quad (\beta > 0),$$

with the dominant steps being Step 5 and Step 6.

5. NUMERICAL EXAMPLE

This algorithm was implemented in ALGEB on a RAVEN 286/10 IBM AT clone at Carleton University. The following example illustrates the calculation of solutions (u, v) to (1.3) in the case

$$n = 9, 198, 968, 367, 101, \quad f = 4, \quad g = 61,$$

using Steps 1–6 as described in §4.

Steps 1, 2. n is composite and the parameters in (4.1) are:

$$p_1 = 12613, \quad a_1 = 1; \quad p_2 = 20333, \quad a_2 = 1; \quad p_3 = 35869, \quad a_3 = 1.$$

Step 3. We have

$$f^{-1} \pmod{n} = 6, 899, 226, 275, 326, \\ m = 6, 899, 226, 275, 341,$$

$$\left(\frac{-m}{12613}\right) = \left(\frac{-m}{20333}\right) = \left(\frac{-m}{35869}\right) = +1.$$

Step 4. The solutions y_i of (4.3) corresponding to the primes p_i , $i = 1, 2, 3$, are

$$y_1 = 4853, \quad y_2 = 9570, \quad y_3 = 14037.$$

Step 5. The four solutions y of $y^2 \equiv -m \pmod{n}$ with $0 < y < n/2$ are

$$\begin{aligned} &382,072,735,980, \\ &1,154,613,726,359, \\ &1,579,334,330,612, \\ &3,116,020,792,951. \end{aligned}$$

Step 6. Applying the Euclidean algorithm to n and each of the solutions y from Step 5 above, we find the corresponding remainders r_k with $r_k < \sqrt{n/f} \leq r_{k-1}$. All four of the values of y give rise to solutions of (1.3) as follows:

| y | r_k | $\sqrt{(n - fr_k^2)}/g$ |
|-------------------|-----------|-------------------------|
| 382,072,735,980 | 717,088 | 342,175 |
| 1,154,613,726,359 | 577,520 | 359,071 |
| 1,579,334,330,612 | 1,376,188 | 163,135 |
| 3,116,020,792,951 | 381,100 | 375,871 |

The values of r_{k-1} and c corresponding to the four solutions above are:

| k | r_{k-1} | c | $(r_{k-1} - cr_k)/g$ |
|-----|------------|-----|----------------------|
| 10 | 26,609,379 | 8 | 342,175 |
| 10 | 25,368,451 | 6 | 359,071 |
| 13 | 55,365,439 | 33 | 163,135 |
| 9 | 24,452,531 | 4 | 375,871 |

in accordance with Theorem 2.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA K1S 5B6 (Hardy and Williams). *E-mail*: kennethhardy@carleton.bitnet; kswilliams@carleton.bitnet

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, BAR-ILAN UNIVERSITY, 52 100 RAMAT-GAN, ISRAEL (Muskat)