

On the evaluation of the Legendre symbol $\left(\frac{A + B\sqrt{m}}{p}\right)$

by

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1. Introduction. Let m be a positive squarefree integer which is either of the form

$$(1.1) \quad m = p_1 p_2 \dots p_r \equiv 1 \pmod{4} \quad (r \geq 1)$$

or of the form

$$(1.2) \quad m = 2p_1 p_2 \dots p_r \equiv 2 \pmod{8} \quad (r \geq 0),$$

where p_1, \dots, p_r are primes congruent to 1 modulo 4. Let (A, B, C) be a triple of positive integers such that

$$(1.3) \quad A^2 = m(B^2 + C^2).$$

(The form of m guarantees that there are infinitely many such triples (A, B, C) .) From (1.3) we see that the greatest common divisor of B and C must divide A and so can be divided out of the equation (1.3). Hence we may assume that

$$(1.4) \quad (A, B) = (A, C) = (B, C) = 1.$$

Let p be an odd prime, not dividing ABC , which is such that

$$(1.5) \quad \begin{cases} \left(\frac{p_i}{p}\right) = 1 & (i = 1, \dots, r), \quad \text{if } m \equiv 1 \pmod{4}, \\ \left(\frac{2}{p}\right) = \left(\frac{p_i}{p}\right) = 1 & (i = 1, \dots, r), \quad \text{if } m \equiv 2 \pmod{8}, \end{cases}$$

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so that for both $m \equiv 1 \pmod{4}$ and $m \equiv 2 \pmod{8}$ we have

$$(1.6) \quad \left(\frac{m}{p}\right) = 1.$$

Hence there exists an integer w such that $w^2 \equiv m \pmod{p}$. Further, by the law of quadratic reciprocity (LQR), we have from (1.5)

$$(1.7) \quad \left(\frac{p}{p_i}\right) = 1 \quad (i = 1, \dots, r).$$

The Dirichlet symbols $\left(\frac{p}{m}\right)_4$ (if $m \equiv 1 \pmod{4}$) and $\left(\frac{p}{m/2}\right)_4$ (if $m \equiv 2 \pmod{8}$) of quartic residuacity are defined by

$$(1.8) \quad \left(\frac{p}{p_1 \dots p_r}\right)_4 = \prod_{i=1}^r \left(\frac{p}{p_i}\right)_4,$$

and for $i = 1, \dots, r$

$$(1.9) \quad \left(\frac{p}{p_i}\right)_4 = \begin{cases} +1, & \text{if } p \text{ is a quartic residue } \pmod{p_i}, \\ -1, & \text{if } p \text{ is a quadratic residue but a quartic nonresidue } \pmod{p_i}. \end{cases}$$

The purpose of this paper is to determine the value of the Legendre symbol $\left(\frac{A+B\sqrt{m}}{p}\right)$, where $\sqrt{m} = \pm w \pmod{p}$. The value of $\left(\frac{A+B\sqrt{m}}{p}\right)$ is independent of the choice of $\pm w$ for \sqrt{m} as

$$\left(\frac{A+B\sqrt{m}}{p}\right)\left(\frac{A-B\sqrt{m}}{p}\right) = \left(\frac{A^2-mB^2}{p}\right) = \left(\frac{mC^2}{p}\right) = 1,$$

so that

$$(1.10) \quad \left(\frac{A+B\sqrt{m}}{p}\right) = \left(\frac{A-B\sqrt{m}}{p}\right).$$

The evaluation of $\left(\frac{A+B\sqrt{m}}{p}\right)$ is carried out in a completely elementary way, requiring nothing more than the manipulation of Jacobi symbols by means of Jacobi's law of quadratic reciprocity, although the method of proof is complicated by the necessity of keeping track of the exact powers of 2 dividing the integers involved. Our theorem provides a unifying result in the theory of rational quartic reciprocity laws as a number of such laws are either special cases of our theorem or can easily be deduced from it. In

particular Scholz's reciprocity law [10] is a special case of our theorem, as are Emma Lehmer's criterion for quartic residuacity ([7], p. 24) and Burde's biquadratic reciprocity law ([4], p. 183). Scholz proved his reciprocity law by means of class field theory. Later authors have given more elementary proofs of it ([5], [9], [12]). Emma Lehmer proved her criterion by studying the rationality of the roots of the period equation of degree 4 considered as a congruence modulo a prime. Burde proved his reciprocity law by means of a study of lattice points. More elementary proofs have been given of Burde's law in [9], [13]. There are many other rational reciprocity results which are also special cases or simple consequences of our theorem (see for example [1], [2], [3], [6], [8], [11]).

After a number of lemmas we prove the following Theorem in Section 2.

THEOREM. *Let m be a positive squarefree integer of the form (1.1) or (1.2). Let A, B, C be positive integers satisfying (1.3) and (1.4), then if $m \equiv 1 \pmod{4}$ we have*

$$\left(\frac{A+B\sqrt{m}}{p}\right) = (-1)^{(p-1)(m-1)/8} \left(\frac{2}{p}\right)^B \left(\frac{p}{m}\right)_4,$$

and if $m \equiv 2 \pmod{8}$ we have

$$\left(\frac{A+B\sqrt{m}}{p}\right) = \begin{cases} (-1)^{(p-1)/8} \left(\frac{p}{m/2}\right)_4, & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{(p+m-1)/8} \left(\frac{p}{m/2}\right)_4, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

We note that the identity

$$2(A+B\sqrt{m})(A+C\sqrt{m}) = (A+B\sqrt{m}+C\sqrt{m})^2$$

gives

$$(1.11) \quad \left(\frac{A+C\sqrt{m}}{p}\right) = \left(\frac{2}{p}\right) \left(\frac{A+B\sqrt{m}}{p}\right).$$

Hence, when $m \equiv 1 \pmod{4}$, in which case B and C are of opposite parity, we may interchange B and C so that B is odd, and to prove the theorem in this case it suffices to show that

$$\left(\frac{A+B\sqrt{m}}{p}\right) = (-1)^{(p-1)(m-1)/8} \left(\frac{2}{p}\right) \left(\frac{p}{m}\right)_4.$$

Throughout the rest of the paper it will be assumed that B is odd when $m \equiv 1 \pmod{4}$.

We now show that Scholz's reciprocity law is a special case of our theorem.

COROLLARY 1 (Scholz, [10]). Let $p \equiv 1 \pmod{4}$ and $q \equiv 1 \pmod{4}$ be distinct primes such that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = 1$. Let ε_p (resp. ε_q) denote the fundamental unit (> 1) of the quadratic field $Q(\sqrt{p})$ (resp. $Q(\sqrt{q})$). Then we have

$$\left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = \left(\frac{\varepsilon_q}{p}\right) = \left(\frac{\varepsilon_p}{q}\right).$$

Proof. In view of the symmetry in p and q , it suffices to prove the first equality. We set

$$\lambda = \begin{cases} 1, & \text{if } q \equiv 1 \pmod{8}, \\ 3, & \text{if } q \equiv 5 \pmod{8}. \end{cases}$$

Then there are positive integers T and U such that

$$\varepsilon_q^\lambda = T + U\sqrt{q}, \quad T \equiv 0 \pmod{2}, \quad U \equiv 1 \pmod{2},$$

and

$$T^2 - qU^2 = -1.$$

Taking $m = q$, $A = qU$, $B = T$, $C = 1$ in the theorem, we obtain

$$\left(\frac{qU + T\sqrt{q}}{p}\right) = \left(\frac{p}{q}\right)_4,$$

that is

$$\left(\frac{\sqrt{q}}{p}\right) \left(\frac{T + U\sqrt{q}}{p}\right) = \left(\frac{p}{q}\right)_4,$$

or

$$\left(\frac{\varepsilon_q}{p}\right) = \left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4$$

as required.

Next we show that Lehmer's criterion is a special case of our theorem.

COROLLARY 2 (Lehmer, [7]). Let $r = a^2 + b^2 = 4n + 1$ (a and b are positive integers with a odd and b even) be a prime, and let q be an odd prime not dividing ab such that $\left(\frac{q}{r}\right) = 1$. Then we have

$$\left(\frac{q}{r}\right)_4 = \left(\frac{(-1)^n 2\lambda(\lambda+1)}{q}\right), \quad \text{where } r \equiv \lambda^2 a^2 \pmod{q}.$$

Proof. Taking $p = q$, $m = r$, $A = r$, $B = a$, $C = b$ in the theorem we obtain

$$\left(\frac{2r + 2a\sqrt{r}}{q}\right) = (-1)^{(q-1)n/2} \left(\frac{q}{r}\right)_4,$$

so that

$$\left(\frac{q}{r}\right)_4 = \left(\frac{(-1)^n(2\lambda^2 a^2 + 2\lambda a^2)}{q}\right) = \left(\frac{(-1)^n 2\lambda(\lambda + 1)}{q}\right)$$

as required.

We remark that Lehmer's criterion ([7], p. 24) included the case $q|ab$. This possibility is not covered by our theorem as we exclude $p|ABC$ from the outset, but is easily treated by the methods employed in this paper.

Finally we show that Burde's reciprocity law also follows from our theorem.

COROLLARY 3 (Burde, [4]). *Let $p \equiv 1 \pmod{4}$ and $q \equiv 1 \pmod{4}$ be distinct primes such that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = 1$. Define positive integers a, b, c, d by*

$$\begin{aligned} p &= a^2 + b^2, & a &\equiv 1 \pmod{2}, & b &\equiv 0 \pmod{2}, \\ q &= c^2 + d^2, & c &\equiv 1 \pmod{2}, & d &\equiv 0 \pmod{2}. \end{aligned}$$

Then

$$\left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = (-1)^{(p-1)/4} \left(\frac{ad-bc}{p}\right).$$

Proof. By the law of quadratic reciprocity we have

$$\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right) = \left(\frac{a^2 + b^2}{a}\right) = \left(\frac{b^2}{a}\right) = 1.$$

Also from the congruence

$$2a(ad-bc)(d + \sqrt{q}) \equiv (ad-bc + a\sqrt{q})^2 \pmod{p}$$

we deduce that

$$\left(\frac{d + \sqrt{q}}{p}\right) = \left(\frac{2}{p}\right) \left(\frac{a}{p}\right) \left(\frac{ad-bc}{p}\right) = (-1)^{(p-1)/4} \left(\frac{ad-bc}{p}\right)$$

Applying the theorem with $m = q$, $A = q$, $B = d$, $C = c$, we obtain

$$\left(\frac{q + d\sqrt{q}}{p}\right) = \left(\frac{p}{q}\right)_4,$$

that is

$$\left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = \left(\frac{d+\sqrt{q}}{p}\right).$$

This completes the proof.

We close this section by noting that from (1.3) we have $m|A^2$, and so, as m is squarefree, we have $m|A$, say,

$$(1.12) \quad A = ma.$$

Then, from (1.3) and (1.4), we obtain

$$(1.13) \quad ma^2 = B^2 + C^2,$$

and

$$(1.14) \quad (a, B) = (a, C) = (B, C) = (m, B) = (m, C) = 1.$$

2. Proof of theorem. In this section we shall prove a number of technical results leading to Lemmas 9 and 10 from which the theorem follows.

LEMMA 1. *Let m be a positive squarefree integer of the form (1.1) or (1.2). Let p be an odd prime satisfying (1.5). Then there exist positive integers e, f, k such that*

$$(2.1) \quad k^2 p = e^2 - mf^2,$$

$$(2.2) \quad (e, f) = (f, k) = (e, k) = (e, p) = (f, p) = (e, m) = (k, m) = 1.$$

Proof. The condition (1.5) guarantees that m is a quadratic residue (mod p) and that p is a quadratic residue (mod m). Hence, by Legendre's theorem, the equation (2.1) is solvable in non-negative integers e, f, k , which are not all zero. Clearly none of e, f, k can be zero so they are in fact all positive. If r is a prime dividing both e and f then $r^2|k^2 p$ and so $r|k$. Thus r is a common factor of e, f and k which can be cancelled throughout the equation (2.1). Hence we may assume without loss of generality that $(e, f) = 1$. It is then easy to check that, as m is squarefree and $p \nmid m$, we have (2.2).

LEMMA 2. *With the notation of Lemma 1 we have the congruences for e, f, k given in Table 1.*

Proof. The congruences for e, f and k follow easily by considering (2.1) modulo 8.

LEMMA 3. *With the notation of Lemma 1, the solution e, f, k of (2.1) and (2.2) can be chosen to satisfy*

$$\begin{cases} f \equiv 0 \pmod{2}, & k \equiv 1 \pmod{2}, & \text{if } p \equiv m \pmod{4}, \\ e \equiv 0 \pmod{2}, & k \equiv 1 \pmod{2}, & \text{if } p \equiv 3 \pmod{4}, m \equiv 1 \pmod{4}. \end{cases}$$

Table 1

$p(\bmod 8)$	$m(\bmod 8)$	congruences for e, f, k
1	1	$e \equiv 1(\bmod 2), f \equiv 0(\bmod 4), k \equiv 1(\bmod 2)$ or $e \equiv f \equiv 1(\bmod 2), k \equiv 0(\bmod 4)$
1	2	$e \equiv 1(\bmod 2), f \equiv 0(\bmod 2), k \equiv 1(\bmod 2)$
1	5	$e \equiv 1(\bmod 2), f \equiv 0(\bmod 4), k \equiv 1(\bmod 2)$ or $e \equiv f \equiv 1(\bmod 2), k \equiv 2(\bmod 4)$
3	1	$e \equiv f \equiv 1(\bmod 2), k \equiv 0(\bmod 4)$ or $e \equiv 2(\bmod 4), f \equiv k \equiv 1(\bmod 2)$
3	2	this case cannot occur
3	5	$e \equiv f \equiv 1(\bmod 2), k \equiv 2(\bmod 4)$ or $e \equiv 0(\bmod 4), f \equiv k \equiv 1(\bmod 2)$
5	1	$e \equiv f \equiv 1(\bmod 2), k \equiv 0(\bmod 4)$ or $e \equiv 1(\bmod 2), f \equiv 2(\bmod 4), k \equiv 1(\bmod 2)$
5	2	this case cannot occur
5	5	$e \equiv f \equiv 1(\bmod 2), k \equiv 2(\bmod 4)$ or $e \equiv 1(\bmod 2), f \equiv 2(\bmod 4), k \equiv 1(\bmod 2)$
7	1	$e \equiv f \equiv 1(\bmod 2), k \equiv 0(\bmod 4)$ or $e \equiv 0(\bmod 4), f \equiv k \equiv 1(\bmod 2)$
7	2	$e \equiv f \equiv k \equiv 1(\bmod 2)$
7	5	$e \equiv f \equiv 1, k \equiv 2(\bmod 4)$ or $e \equiv 2(\bmod 4), f \equiv k \equiv 1(\bmod 2)$

Proof. If $p \equiv m(\bmod 4)$ and e, f, k is a solution of (2.1) and (2.2) with $f \equiv 1(\bmod 2)$ and $k \equiv 0(\bmod 2)$ then e', f', k' defined by

$$e' = e \left(\frac{p+m}{2} \right), \quad f' = \left| \frac{(p-m)}{2} f - pk \right|, \quad k' = \left| \frac{(p-m)}{2} k + mf \right|$$

is a solution in positive integers of (2.1) with

$$f' \equiv 0 \pmod{2}, \quad k' \equiv 1 \pmod{2}.$$

Dividing each of e', f', k' by their G.C.D. we obtain the required solution.

If $p \equiv 3 \pmod{4}, m \equiv 1 \pmod{4}$ let e, f, k be a solution in positive integers of (2.1) and (2.2) with $e \equiv 1 \pmod{2}, k \equiv 0 \pmod{2}$. Let X, Y be the smallest solution in positive integers of $X^2 - pY^2 = 1$. Then it is well known and easily verified that

$$X \equiv 0 \pmod{2}, \quad Y \equiv 1 \pmod{2}.$$

Replacing e, f, k by $e' = eX + pkY, f' = f, k' = kX + eY$, we obtain a solution in positive integers of (2.1) with $e' \equiv 0 \pmod{2}, k' \equiv 1 \pmod{2}$. Dividing each of e', f', k' by their G.C.D. we obtain the required solution.

In view of Lemmas 2 and 3 and the choice $B \equiv 1 \pmod{2}$ it suffices to consider the following cases.

Table 2

Case	$p \pmod{8}$	$m \pmod{8}$	congruences for A, B, C	congruences for e, f, k
I	1	1	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 0 \pmod{4}$	$e \equiv 1 \pmod{2}, f \equiv 0 \pmod{4},$ $k \equiv 1 \pmod{2}$
II	1	2	$A \equiv 2 \pmod{4},$ $B \equiv C \equiv 1 \pmod{2}$	$e \equiv 1 \pmod{2}, f \equiv 0 \pmod{2},$ $k \equiv 1 \pmod{2}$
III	1	5	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 2 \pmod{4}$	$e \equiv 1 \pmod{2}, f \equiv 0 \pmod{4},$ $k \equiv 1 \pmod{2}$
IV	3	1	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 0 \pmod{4}$	$e \equiv 2 \pmod{4}, f \equiv k \equiv 1 \pmod{2}$
V	3	5	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 2 \pmod{4}$	$e \equiv 0 \pmod{4}, f \equiv k \equiv 1 \pmod{2}$
VI	5	1	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 0 \pmod{4}$	$e \equiv 1 \pmod{2}, f \equiv 2 \pmod{4},$ $k \equiv 1 \pmod{2}$
VII	5	5	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 2 \pmod{4}$	$e \equiv 1 \pmod{2}, f \equiv 2 \pmod{4},$ $k \equiv 1 \pmod{2}$
VIII	7	1	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 0 \pmod{4}$	$e \equiv 0 \pmod{4}, f \equiv k \equiv 1 \pmod{2}$
IX	7	2	$A \equiv 2 \pmod{4},$ $B \equiv C \equiv 1 \pmod{2}$	$e \equiv f \equiv k \equiv 1 \pmod{2}$
X	7	5	$A \equiv B \equiv 1 \pmod{2},$ $C \equiv 2 \pmod{4}$	$e \equiv 2 \pmod{4}, f \equiv k \equiv 1 \pmod{2}$

The following simple observations will be important in what follows:

(2.3) $a \equiv B \equiv k \equiv 1 \pmod{2},$

(2.4) $Af + Be \equiv 1 \pmod{2},$

(2.5) $ae + Bf \equiv \begin{cases} 0 \pmod{2}, & \text{in case IX,} \\ 1 \pmod{2}, & \text{otherwise.} \end{cases}$

It is convenient at this point to introduce the following notation: if n is a positive integer the largest odd divisor of n is denoted by n^* . In particular we set

$$(2.6) \quad e = 2^\varepsilon e^*, \quad \varepsilon \geq 0, \quad e^* \equiv 1 \pmod{2},$$

$$(2.7) \quad f = 2^\zeta f^*, \quad \zeta \geq 0, \quad f^* \equiv 1 \pmod{2},$$

$$(2.8) \quad m = 2^\mu m^*, \quad \mu = 0 \text{ or } 1, \quad m^* \equiv 1 \pmod{4}.$$

We note that

$$(2.9) \quad \varepsilon = 0, \quad \text{if } p \equiv 1 \pmod{4},$$

$$(2.10) \quad \zeta = 0, \quad \text{if } p \equiv 3 \pmod{4},$$

$$(2.11) \quad \mu = \begin{cases} 0, & \text{if } m \equiv 1 \pmod{4}, \\ 1, & \text{if } m \equiv 2 \pmod{8}. \end{cases}$$

LEMMA 4. *With the notation of Lemma 1 and Table 2, we have*

$$(2.12) \quad \left(\frac{e}{p}\right) = \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{ke^*}\right)^\mu \left(\frac{p}{m^*}\right)_4$$

and

$$(2.13) \quad \left(\frac{f}{p}\right) = \left(\frac{-1}{f^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\zeta.$$

Proof. We have

$$\begin{aligned} \left(\frac{e}{p}\right) &= \left(\frac{2}{p}\right)^\varepsilon \left(\frac{e^*}{p}\right) && \text{(by (2.6))} \\ &= \left(\frac{2}{p}\right)^\varepsilon \left(\frac{-1}{e^*}\right)^{(p-1)/2} \left(\frac{p}{e^*}\right) && \text{(by LQR)} \\ &= \left(\frac{-1}{e^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\varepsilon \left(\frac{k^2 p}{e^*}\right) && \text{(by (2.2))} \\ &= \left(\frac{-1}{e^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\varepsilon \left(\frac{e^2 - mf^2}{e^*}\right) && \text{(by (2.1))} \\ &= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{p}\right)^\varepsilon \left(\frac{m}{e^*}\right) && \text{(by (2.2))} \\ &= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{p}\right)^\varepsilon \left(\frac{2}{e^*}\right)^\mu \left(\frac{m^*}{e^*}\right) && \text{(by (2.8))} \\ &= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{p}\right)^\varepsilon \left(\frac{2}{e^*}\right)^\mu \left(\frac{e^*}{m^*}\right) && \text{(by LQR)} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{e^*}\right)^\mu \left(\frac{e}{m^*}\right) && \text{(by (2.6))} \\
&= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{e^*}\right)^\mu \left(\frac{e^2}{m^*}\right)_4 && \text{(by (1.1), (1.2))} \\
&= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{e^*}\right)^\mu \left(\frac{k^2 p}{m^*}\right)_4 && \text{(by (2.1))} \\
&= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{e^*}\right)^\mu \left(\frac{k}{m^*}\right) \left(\frac{p}{m^*}\right)_4 \\
&= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{e^*}\right)^\mu \left(\frac{m^*}{k}\right) \left(\frac{p}{m^*}\right)_4 && \text{(by LQR)} \\
&= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{ke^*}\right)^\mu \left(\frac{m}{k}\right) \left(\frac{p}{m^*}\right) && \text{(by (2.8))} \\
&= \left(\frac{-1}{e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\varepsilon \left(\frac{2}{ke^*}\right)^\mu \left(\frac{p}{m^*}\right)_4,
\end{aligned}$$

as

$$\left(\frac{m}{k}\right) = \left(\frac{mf^2}{k}\right) = \left(\frac{e^2 - k^2 p}{k}\right) = \left(\frac{e^2}{k}\right) = 1.$$

This completes the proof of (2.12).

Next we have

$$\begin{aligned}
\left(\frac{f}{p}\right) &= \left(\frac{2}{p}\right)^\zeta \left(\frac{f^*}{p}\right) && \text{(by (2.7))} \\
&= \left(\frac{2}{p}\right)^\zeta \left(\frac{-1}{f^*}\right)^{(p-1)/2} \left(\frac{p}{f^*}\right) && \text{(by LQR)} \\
&= \left(\frac{-1}{f^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\zeta \left(\frac{k^2 p}{f^*}\right) && \text{(by (2.2))} \\
&= \left(\frac{-1}{f^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\zeta \left(\frac{e^2 - mf^2}{f^*}\right) && \text{(by (2.1))} \\
&= \left(\frac{-1}{f^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\zeta \left(\frac{e^2}{f^*}\right) \\
&= \left(\frac{-1}{f^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\zeta
\end{aligned}$$

which completes the proof of (2.13).

LEMMA 5. With the notation of Lemma 1 and Table 2, we have

$$(2.14) \quad \left(\frac{B}{m^*}\right) = \left(\frac{2}{B}\right)^\mu,$$

$$(2.15) \quad \left(\frac{k}{m^*}\right) = \left(\frac{2}{k}\right)^\mu,$$

$$(2.16) \quad \left(\frac{e}{m^*}\right) = \left(\frac{2}{k}\right)^\mu \left(\frac{p}{m^*}\right)_4.$$

Proof. We have

$$\begin{aligned} \left(\frac{B}{m^*}\right) &= \left(\frac{m^*}{B}\right) && \text{(by LQR)} \\ &= \left(\frac{2}{B}\right)^\mu \left(\frac{m}{B}\right) && \text{(by (2.8))} \\ &= \left(\frac{2}{B}\right)^\mu \left(\frac{ma^2}{B}\right) && \text{(by (1.14))} \\ &= \left(\frac{2}{B}\right)^\mu \left(\frac{B^2 + C^2}{B}\right) && \text{(by (1.13))} \\ &= \left(\frac{2}{B}\right)^\mu. \end{aligned}$$

This completes the proof of (2.14).

Next we have

$$\begin{aligned} \left(\frac{k}{m^*}\right) &= \left(\frac{m^*}{k}\right) && \text{(by LQR)} \\ &= \left(\frac{2}{k}\right)^\mu \left(\frac{m}{k}\right) && \text{(by (2.8))} \\ &= \left(\frac{2}{k}\right)^\mu \left(\frac{mf^2}{k}\right) && \text{(by (2.2))} \\ &= \left(\frac{2}{k}\right)^\mu \left(\frac{e^2 - k^2 p}{k}\right) && \text{(by (2.1))} \\ &= \left(\frac{2}{k}\right)^\mu. \end{aligned}$$

This completes the proof of (2.15).

Finally we have

$$\begin{aligned}
 \left(\frac{e}{m^*}\right) &= \left(\frac{e^2}{m^*}\right) \\
 &= \left(\frac{e^2 - mf^2}{m^*}\right)_4 \\
 &= \left(\frac{k^2 p}{m^*}\right)_4 \quad (\text{by (2.1)}) \\
 &= \left(\frac{k}{m^*}\right) \left(\frac{p}{m^*}\right)_4 \\
 &= \left(\frac{2}{k}\right)^\mu \left(\frac{p}{m^*}\right)_4 \quad (\text{by (2.15)}).
 \end{aligned}$$

This completes the proof of (2.16).

LEMMA 6. *With the notation of Lemma 1 and Table 2, we have*

$$\left(\frac{Af + Be}{m^*}\right) = \left(\frac{2}{Bk}\right)^\mu \left(\frac{p}{m^*}\right)_4.$$

Proof. This follows immediately from Lemma 5 as

$$\left(\frac{Af + Be}{m^*}\right) = \left(\frac{maf + Be}{m^*}\right) = \left(\frac{Be}{m^*}\right) = \left(\frac{B}{m^*}\right) \left(\frac{e}{m^*}\right).$$

LEMMA 7. *With the notation of Lemma 1 and Table 2, we have*

$$(2.17) \quad (Af + Be, ae + Bf) = l^2,$$

where l is an odd positive integer such that

$$(2.18) \quad (l, maBefp) = 1.$$

Proof. We first recall from (2.4) that $Af + Be$ is odd. Therefore $(Af + Be, ae + Bf)$ is odd. Suppose q is an odd prime such that

$$q^{2h+1} \parallel (Af + Be, ae + Bf).$$

From

$$(2.19) \quad (Af + Be)^2 - m(ae + Bf)^2 = (B^2 - ma^2)(e^2 - mf^2) = -C^2 k^2 p,$$

we have $q^{4h+2} \mid C^2 k^2 p$, so $q^{2h+1} \mid Ck$. Hence either $q^{h+1} \mid C$ or $q^{h+1} \mid k$.

Suppose first that $q^{h+1} \mid C$. It is easy to check that $(q, aBmp) = 1$. Further as $(e, f) = 1$ and $q^{2h+1} \mid ae + Bf$ we have $(q, ef) = 1$. Then eliminating B from the congruences

$$Be \equiv -maf \pmod{q^{2h+1}}, \quad Bf \equiv -ae \pmod{q^{2h+1}}$$

we obtain $a(e^2 - mf^2) \equiv 0 \pmod{q^{2h+1}}$, so that $k^2 p \equiv 0 \pmod{q^{2h+1}}$, giving $q^{h+1} | k$.

Suppose now that $q^{h+1} | k$. It is easy to check that $(q, efm) = 1$. Further as $(a, B) = 1$ and $q^{2h+1} | ae + Bf$ we have $(q, aB) = 1$. Then eliminating f from the congruences

$$maf = -Be \pmod{q^{2h+1}}, \quad Bf \equiv -ae \pmod{q^{2h+1}},$$

we obtain $e(ma^2 - B^2) \equiv 0 \pmod{q^{2h+1}}$, so that $C^2 \equiv 0 \pmod{q^{2h+1}}$, giving $q^{h+1} | C$, $(q, p) = 1$.

Hence in both cases we have $q^{h+1} | C$, $q^{h+1} | k$, and $(q, aBmefp) = 1$. Then from $ma^2 - B^2 = C^2 \equiv 0 \pmod{q^{2h+2}}$ and $e^2 - mf^2 = k^2 p \equiv 0 \pmod{q^{2h+2}}$, we obtain on eliminating m

$$a^2 e^2 - B^2 f^2 \equiv 0 \pmod{q^{2h+2}},$$

so that

$$(ae + Bf)(ae - Bf) \equiv 0 \pmod{q^{2h+2}}.$$

Further we have

$$\begin{aligned} a^2 m^2 f^2 - B^2 e^2 &\equiv mf^2(B^2 + C^2) - B^2 e^2 \pmod{q^{2h+2}} \\ &\equiv -B^2 k^2 p \pmod{q^{2h+2}} \\ &\equiv 0 \pmod{q^{2h+2}}, \end{aligned}$$

so that

$$(amf + Be)(amf - Be) \equiv 0 \pmod{q^{2h+2}}.$$

As $(q, aBmef) = 1$ q cannot divide both of $ae + Bf$, $ae - Bf$ nor both of $amf + Be$, $amf - Be$. Since q divides $amf + Be$ and $ae + Bf$ we must have

$$q^{2h+2} | amf + Be, \quad q^{2h+2} | ae + Bf.$$

This contradicts the definition of h . Thus every odd prime divisor of $(Af + Be, ae + Bf)$ must divide $(Af + Be, ae + Bf)$ to an even power, so that as $(Af + Be, ae + Bf)$ is odd we have

$$(Af + Be, ae + Bf) = l^2,$$

where l is an odd positive integer. It is easy to check that $(l, maBefp) = 1$. We just show that $(l, p) = 1$. For suppose $p | l$ then we have $p | Af + Be$, giving

$$\begin{aligned} mC^2 f^2 &\equiv mC^2 f^2 - B^2(e^2 - mf^2) \pmod{p} \\ &\equiv A^2 f^2 - B^2 e^2 \pmod{p} \\ &\equiv 0 \pmod{p}, \end{aligned}$$

which is impossible as $p \nmid m$, $p \nmid C$, $p \nmid f$.

This completes the proof of Lemma 7.

LEMMA 8. *With the notation of Lemmas 1, 7 and Table 2, we set*

$$(2.20) \quad Af + Be = l^2 t, \quad ae + Bf = l^2 u,$$

so that t and u are positive integers such that

$$(2.21) \quad (t, u) = 1$$

and

$$(2.22) \quad t \equiv 1 \pmod{2}, \quad u \equiv \begin{cases} 0 \pmod{2}, & \text{in case IX,} \\ 1 \pmod{2}, & \text{otherwise.} \end{cases}$$

Then there exists a positive integer v such that

$$(2.23) \quad Ck = l^2 v,$$

$$(2.24) \quad t^2 - mu^2 = -pv^2.$$

Moreover setting

$$(2.25) \quad u = 2^\theta u^*, \quad u^* \equiv 1 \pmod{2}, \quad \theta \geq 0,$$

so that $\theta = 0$ except in case IX, we have

$$(2.26) \quad \left(\frac{Af + Be}{p} \right) = \left(\frac{-1}{t} \right)^{(p-1)/2} \left(\frac{2}{Bkt} \right)^\mu \left(\frac{p}{m^*} \right)_4$$

and

$$(2.27) \quad \left(\frac{ae + Bf}{p} \right) = \left(\frac{-1}{u^*} \right)^{(p+1)/2}.$$

Proof. Appealing to (2.19) and (2.20) we obtain

$$(2.28) \quad l^4(t^2 - mu^2) = -C^2 k^2 p.$$

As p is an odd prime, which is coprime with l , we have $l^2 | Ck$, so that there is a positive integer v such that $Ck = l^2 v$, which is (2.23). Then from (2.29) we obtain $t^2 - mu^2 = -pv^2$, which is (2.24). As $(t, u) = 1$ and m is squarefree, it is easy to verify that

$$(2.29) \quad (t, v) = (t, p) = (u, v) = (t, m) = (v, m) = (u, p) = 1.$$

Next we have

$$\left(\frac{Af + Be}{p} \right) = \left(\frac{l^2 t}{p} \right) \quad \text{(by (2.20))}$$

$$= \left(\frac{t}{p} \right) \quad \text{(by (2.18))}$$

$$= \left(\frac{-1}{t} \right)^{(p-1)/2} \left(\frac{p}{t} \right) \quad \text{(by LQR)}$$

$$\begin{aligned}
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{pv^2}{t}\right) && \text{(by (2.29))} \\
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{mu^2 - t^2}{t}\right) && \text{(by (2.24))} \\
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{m}{t}\right) && \text{(by (2.21))} \\
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{2}{t}\right)^\mu \left(\frac{m^*}{t}\right) && \text{(by (2.8))} \\
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{2}{t}\right)^\mu \left(\frac{t}{m^*}\right) && \text{(by LQR)} \\
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{2}{t}\right)^\mu \left(\frac{l^2 t}{m^*}\right) && \text{(by (2.18))} \\
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{2}{t}\right)^\mu \left(\frac{Af + Be}{m^*}\right) && \text{(by (2.20))} \\
 &= \left(\frac{-1}{t}\right)^{(p-1)/2} \left(\frac{2}{Bkt}\right)^\mu \left(\frac{p}{m^*}\right)_4 && \text{(by Lemma 6).}
 \end{aligned}$$

This completes the proof of (2.26).

Also we have

$$\begin{aligned}
 \left(\frac{ae + Bf}{p}\right) &= \left(\frac{l^2 u}{p}\right) && \text{(by (2.20))} \\
 &= \left(\frac{2}{p}\right)^\theta \left(\frac{u^*}{p}\right) && \text{(by (2.18), (2.25))} \\
 &= \left(\frac{2}{p}\right)^\theta \left(\frac{-1}{u^*}\right)^{(p-1)/2} \left(\frac{p}{u^*}\right) && \text{(by LQR)} \\
 &= \left(\frac{2}{p}\right)^\theta \left(\frac{-1}{u^*}\right)^{(p-1)/2} \left(\frac{pv^2}{u^*}\right) && \text{(by (2.29))} \\
 &= \left(\frac{2}{p}\right)^\theta \left(\frac{-1}{u^*}\right)^{(p-1)/2} \left(\frac{mu^2 - t^2}{u^*}\right) && \text{(by (2.24))} \\
 &= \left(\frac{2}{p}\right)^\theta \left(\frac{-1}{u^*}\right)^{(p+1)/2} \\
 &= \left(\frac{-1}{u^*}\right)^{(p+1)/2}
 \end{aligned}$$

This completes the proof of (2.27).

LEMMA 9. With the notation of Lemmas 1, 7, 8 and Table 2, we have

$$(2.30) \quad \left(\frac{A+B\sqrt{m}}{p}\right) = \left(\frac{-1}{tf^*}\right)^{(p-1)/2} \left(\frac{2}{p}\right)^\zeta \left(\frac{2}{Bkt}\right)^\mu \left(\frac{p}{m^*}\right)_4$$

and

$$(2.31) \quad \left(\frac{A+B\sqrt{m}}{p}\right) = \left(\frac{-1}{u^*e^*}\right)^{(p+1)/2} \left(\frac{2}{pm^*}\right)^\epsilon \left(\frac{2}{ke^*}\right)^\mu \left(\frac{p}{m^*}\right)_4.$$

Proof. Equation (2.30) follows from the equation

$$\left(\frac{f}{p}\right) \left(\frac{A+B\sqrt{m}}{p}\right) = \left(\frac{Af+Be}{p}\right)$$

by appealing to (2.13) and (2.26).

Equation (2.31) follows from the equation

$$\left(\frac{e}{p}\right) \left(\frac{A+B\sqrt{m}}{p}\right) = \left(\frac{ae+Bf}{p}\right)$$

by appealing to (2.12) and (2.27).

LEMMA 10. With the notation of Lemmas 1, 7, 8 and Table 2, in case II we have

$$\left(\frac{2}{Bkt}\right) = (-1)^{(p-1)/8},$$

and in case IX we have

$$\left(\frac{2}{ke}\right) = (-1)^{(p+m-1)/8}.$$

Proof. In case II we have

$$\left(\frac{2}{Bet}\right) = \left(\frac{2}{Bel^2t}\right) = \left(\frac{2}{Be(Af+Be)}\right) = \left(\frac{2}{ABef+B^2e^2}\right) = \left(\frac{2}{ABef+1}\right) = (-1)^{f/2},$$

and

$$\begin{aligned} \left(\frac{2}{ke}\right) &= \left(\frac{2}{k}\right)^{-p} \left(\frac{2}{e}\right) = (-1)^{-(k^2-1)p+(e^2-1)/8} \\ &= (-1)^{mf^2/8+(p-1)/8} = (-1)^{f/2+(p-1)/8}, \end{aligned}$$

so

$$\left(\frac{2}{Bkt}\right) = \left(\frac{2}{Bet}\right) \left(\frac{2}{ke}\right) = (-1)^{(p-1)/8}$$

as required.

In case IX, similarly to above, we have

$$\left(\frac{2}{ke}\right) = (-1)^{(p-1+mf^2)/8} = (-1)^{(p+m-1)/8}$$

as required.

Proof of theorem. The theorem now follows by a case by case examination from (2.30) of Lemma 9 when $p \equiv 1 \pmod{4}$ and from (2.31) of Lemma 9 when $p \equiv 3 \pmod{4}$ together with Lemma 10 in Cases II and IX.

3. Tables. A DEC Professional 350 minicomputer was programmed to calculate $A, B, C, e, f, k, \left(\frac{A+B\sqrt{m}}{p}\right), \left(\frac{p}{m^*}\right)_4$, etc. for a variety of values of $m (< 200)$ and primes $p (< 150)$. Tables of these values were used effectively in checking results and in formulating the correct form of the theorem. For the convenience of the reader a short list of these values is given below.

Table 3

m	A	B	C	p	$\left(\frac{A+B\sqrt{m}}{p}\right)$	$(-1)^{(p-1)(m-1)/8}$	$\left(\frac{2}{p}\right)^B$	$\left(\frac{m}{p}\right)_4$
5	25	2	11	19	1	-1	1	-1
5	65	2	29	11	-1	-1	1	1
13	65	1	18	17	-1	1	1	-1
13	65	6	17	23	1	-1	1	-1
17	17	1	4	13	-1	1	-1	1
17	17	4	1	13	1	1	1	1
29	29	2	5	7	-1	-1	1	1
29	29	5	2	7	-1	-1	1	1
37	37	1	6	7	-1	-1	1	1
37	37	6	1	7	-1	-1	1	1
41	41	4	5	23	1	1	1	1
41	41	5	4	23	1	1	1	1
53	53	2	7	11	1	-1	1	-1
53	53	7	2	11	-1	-1	-1	-1
61	61	5	6	13	-1	1	-1	1
61	61	6	5	13	1	1	1	1
65	65	1	8	29	1	1	-1	-1
65	65	4	7	29	-1	1	1	-1
73	73	3	8	19	1	1	-1	-1
73	73	8	3	19	-1	1	1	-1
85	85	2	9	19	-1	-1	1	1
85	85	6	7	19	-1	-1	1	1
89	89	5	8	11	-1	1	-1	1
89	89	8	5	11	1	1	1	1
97	97	4	9	11	-1	1	1	-1
97	97	9	4	11	1	1	-1	-1

m	A	B	C	p	$\left(\frac{A+B\sqrt{m}}{p}\right)$	$(-1)^{(p-1)/8}$, if $p \equiv 1(8)$ $(-1)^{(p+m-1)/8}$, if $p \equiv 7(8)$	$\left(\frac{p}{m/2}\right)_4$
10	50	9	13	31	-1	-1	1
10	50	13	9	31	-1	-1	1
26	130	11	23	17	-1	1	-1
26	130	17	19	23	-1	1	-1
34	34	3	5	47	1	1	1
34	170	3	29	47	1	1	1
58	58	3	7	23	1	1	1
58	58	7	3	23	1	1	1
74	74	5	7	41	1	-1	-1
74	74	7	5	41	1	-1	-1
82	82	1	9	23	-1	-1	1
82	82	9	1	23	-1	-1	1

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