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An application of dihedral fields to representations of primes by binary quadratic forms

by

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1. Introduction. Let $H(m)$ denote the strict ideal class group of the quadratic field $Q(\sqrt{m})$ of discriminant m. We have

$$H(m) \simeq Z_{2^{n_1}} \times Z_{2^{n_2}} \times \ldots \times Z_{2^{n_k}} \times G, \tag{1.1}$$

where the order g of the group G is odd and Z_{2^n} denotes the cyclic group of order 2^n.

Let p be a prime number such that $\left(\frac{m}{p}\right) = 1$. Then p is represented by two inverse classes C_p, C_p^{-1} (or one ambiguous class) of binary quadratic forms of discriminant m. Gauss's theory of genera determines C_p modulo squares in the composition class group of discriminant m.

In this paper we determine the class C_p modulo fourth powers in the simplest case, namely when

$$H(m) \simeq Z_{2^n} \times G, \qquad n \geqslant 2, \tag{1.2}$$

and the class C_p is a square, that is p is a prime on which all the generic characters have the value $+1$. It is known (see for example [2]) that (1.2) occurs precisely for the following values of the discriminant m:

(I) $m = -4r$, $r\,(\text{prime}) \equiv 1 \pmod 8$;

(II) $m = -8r$, $r\,(\text{prime}) \equiv 1 \pmod 8$;

(III) $m = -8q$, $q\,(\text{prime}) \equiv 7 \pmod 8$;

(IV) $m = -qr$, $q\,(\text{prime}) \equiv 3 \pmod 4$, $r\,(\text{prime}) \equiv 1 \pmod 4$, $\left(\frac{q}{r}\right) = 1$;

(V) $m = 8r$, $r\,(\text{prime}) \equiv 1 \pmod 8$;

(VI) $m = qr$, $q\,(\text{prime}) \equiv r\,(\text{prime}) \equiv 1 \pmod 4$, $\left(\frac{q}{r}\right) = 1$.

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We define
$q = 1$ in case (I),
$q = 2$ in cases (II), (V),
$r = 2$ in case (III)
and

$$k_q = \begin{cases} Q(\sqrt{-q}) & \text{in cases (I), (II), (III), (IV);} \\ Q(\sqrt{q}) & \text{in cases (V), (VI);} \end{cases}$$

$$k_r = Q(\sqrt{r}); \quad k_m = Q(\sqrt{m})$$

$$K = Q(\sqrt{r}, \sqrt{m}) = \begin{cases} Q(\sqrt{r}, \sqrt{-q}) & \text{in cases (I) to (IV),} \\ Q(\sqrt{r}, \sqrt{q}) & \text{in cases (V), (VI).} \end{cases}$$

The strict class number of the quadratic field $Q(\sqrt{d})$ will be denoted by $h(d)$.

Throughout this paper the symbol $\left(\frac{x+y\sqrt{n}}{p}\right)$, where n and x^2-ny^2 are quadratic residues of the odd prime p, will be used both as a Legendre symbol, in which case $\sqrt{n}$ is interpreted as a rational integer modulo p, as well as (equivalently) the quadratic residue symbol $\left[\frac{x+y\sqrt{n}}{P}\right]_2$ in the ring of integers of $Q(\sqrt{n})$, where P is either of the two prime ideals dividing p.
We prove:

THEOREM 1. *Let r be a prime $\equiv 1 \pmod 8$ and p a prime satisfying $\left(\frac{-1}{p}\right) = \left(\frac{p}{r}\right) = 1$, so that p is represented by the classes C_p and C_p^{-1} of discriminant $-4r$, and there exist integers a, b, e and f such that*

$$p = a^2+b^2, \tag{1.3}$$

$$p^{h(r)} = e^2-rf^2, \quad e > 0, \quad (e, f) = 1. \tag{1.4}$$

Then the class C_p is a fourth power if, and only if, for any solutions of (1.3) *and* (1.4), $\left(\frac{a+b\sqrt{-1}}{r}\right) = 1$ *or, equivalently,* $e+f \equiv 1 \pmod 4$.

THEOREM 2. *Let r be a prime $\equiv 1 \pmod 8$ and p a prime satisfying $\left(\frac{-2}{p}\right) = \left(\frac{p}{r}\right) = 1$, so that p is represented by the classes C_p and C_p^{-1} of discriminant $-8r$, and there exist integers a, b, e and f such that*

(1.5) $$p = a^2 + 2b^2,$$

(1.6) $$p^{h(r)} = e^2 - rf^2, \quad e > 0, \quad (e, f) = 1.$$

Then the class C_p *is a fourth power if, and only if, for any solutions of* (1.5) *and* (1.6), $\left(\frac{a+b\sqrt{-2}}{r}\right) = 1$ *or, equivalently,* $\left(\frac{2}{p}\right)^{(r-1)/8}\left(\frac{-2}{e+f}\right) = 1$.

THEOREM 3. *Let* $q \equiv 7 \pmod 8$ *be a prime. Let* p *be a prime satisfying* $\left(\frac{p}{q}\right) = \left(\frac{2}{p}\right) = 1$, *so that* p *is represented by the classes* C_p *and* C_p^{-1} *of discriminant* $-8q$, *and there exist integers* a, b, e *and* f *such that*

(1.7) $$p^{h(-q)} = a^2 + qb^2, \quad (a, b) = 1,\ a \text{ or } b \equiv 1 \pmod 4,$$

(1.8) $$p = e^2 - 2f^2, \quad e > 0.$$

Then the class C_p *is a fourth power if, and only if, for any solutions of* (1.7) *and* (1.8),

$$\left(\frac{-1}{p}\right)^{(q+1)/8}\left(\frac{2}{a+b}\right) = 1 \quad \text{or, equivalently,} \quad \left(\frac{e+f\sqrt{2}}{q}\right) = 1.$$

We note that Theorem 3 of [1] is part of the special case $q = 7$ of our Theorem 3.

THEOREM 4. *Let* $q \equiv 3 \pmod 4$ *and* $r \equiv 1 \pmod 4$ *be primes such that* $\left(\frac{q}{r}\right) = 1$. *Let* p *be a prime satisfying* $\left(\frac{p}{q}\right) = \left(\frac{p}{r}\right) = 1$, *so that* p *is represented by the classes* C_p *and* C_p^{-1} *of discriminant* $-qr$ *and there exist integers* a, b, e *and* f *such that*

(1.9) $$4p^{h(-q)} = a^2 + qb^2, \quad (a, b) = 1 \text{ or } 2,$$

(1.10) $$4p^{h(r)} = e^2 - rf^2, \quad (e, f) = 1 \text{ or } 2, \quad e > 0.$$

Then the class C_p *is a fourth power if, and only if, for any solutions of* (1.9) *and* (1.10),

$$\left(\frac{(a+b\sqrt{-q})/2}{r}\right) = 1 \quad \text{or, equivalently,} \quad \left(\frac{(e+f\sqrt{r})/2}{q}\right) = 1.$$

We note that Theorems 6 and 7 of [1] can be deduced as special cases of our Theorem 4 with $q = 3$, $r = 13$ and $q = 11$, $r = 5$, respectively.

THEOREM 5. *Let* r *be a prime* $\equiv 1 \pmod 8$ *and* p *be a prime satisfying* $\left(\frac{2}{p}\right)$

$=\left(\frac{p}{r}\right)=1$, *so that p is represented by the classes C_p and C_p^{-1} of discriminant $8p$, and that there exist integers a, b, e and f such that*

$$p=a^2-2b^2, \quad (a, b)=1, \quad a>0; \tag{1.11}$$

$$p^{h(r)}=e^2-rf^2, \quad (e, f)=1, \quad e+f\equiv 1 \pmod 4. \tag{1.12}$$

Then C_p is a fourth power if, and only if, for any solutions of (1.11) *and* (1.12), $\left(\frac{a+b\sqrt{2}}{r}\right)=1$ *or, equivalently,* $e+f\equiv 1 \pmod 8$.

THEOREM 6. *Let q and r be primes $\equiv 1 \pmod 4$ such that $\left(\frac{q}{r}\right)=1$. Let p be a prime satisfying $\left(\frac{p}{q}\right)=\left(\frac{p}{r}\right)=1$, so that p is represented by the classes C_p and C_p^{-1} of discriminant qr and that there exist integers a, b, e and f such that*

$$4p^{h(q)}=a^2-qb^2, \quad (a, b)=1 \text{ or } 2; \tag{1.13}$$

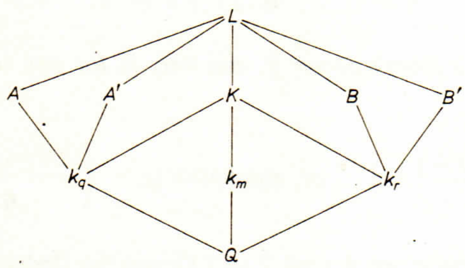
$$4p^{h(r)}=e^2-rf^2, \quad (e, f)=1 \text{ or } 2. \tag{1.14}$$

Then C_p is a fourth power if, and only if, for any solutions of (1.13) *and* (1.14),

$$\left(\frac{(a+b\sqrt{q})/2}{r}\right)=1 \quad \text{or, equivalently,} \quad \left(\frac{(e+f\sqrt{r})/2}{q}\right)=1.$$

2. Proof of the theorems. The assumption (1.2) implies that the strict class group of k_m contains exactly one subgroup of index 4. Let L be the extension of k_m corresponding to this subgroup by class field theory. Then L is the cyclic extension of degree 4 of k_m, unramified at any finite prime.

It is known ([3]) that L is a dihedral extension of Q whose quadratic subfields are k_m, k_q and k_r and whose quartic subfields are the field K, two fields A and A' containing k_q but neither k_r nor k_m, and two fields B and B' containing k_r but neither k_q nor k_m.



Let p be a prime on which all the generic characters of k_m take the value $+1$. Then p is completely decomposed in K, the genus field of k_m, and the classes C_p, C_p^{-1} are squares. The classes C_p, C_p^{-1} are fourth powers if, and only if p is completely decomposed in L, that is if p is completely decomposed in any of the four fields A, A', B or B'.

Consider for instance the extension B/k_r, of conductor f_B. There exists a character χ_B of order 2 on the group of ideals of k_r prime to f_B such that a prime ideal i of k_r is decomposed in B if, and only if, $\chi_B(i) = 1$. The value $\chi_B(i)$ is equal to $\chi_B(i^{h(r)})$, as $h(r)$ is odd, and the value of χ_B on principal ideals prime to f_B has been calculated in Propositions 2.6 to 2.11 of [4]. Applying this to either of the ideals $\bar{p}_1$, $\bar{p}_2$ such that $(p) = \bar{p}_1 \bar{p}_2$ in k_r we shall obtain the results for those theorems involving the integers e and f. The results involving the integers a and b will be obtained by considering the extension A/k_q. We give the details of the proof of Theorem 3, the other proofs are similar. In this case the decompositions of p, q and $r = 2$ in the fields k_q and k_r are the following:

$$(p) = p_1 p_2, \quad (q) = (\sqrt{-q})^2, \quad (2) = r_1 r_2 \quad \text{in } k_q, \tag{2.1}$$

$$(p) = \bar{p}_1 \bar{p}_2, \quad (q) = \bar{q}_1 \bar{q}_2, \quad 2 = (\sqrt{2})^2 \quad \text{in } k_r. \tag{2.2}$$

We first consider the extension A/k_q. By Section 2 of [4] one of r_1, r_2 is ramified in A/k_q and the other in A'/k_q; we choose the notation so that r_1 ramifies in A/k_q. By Proposition 2.9 of [4] the conductor of A/k_q is r_1^3 and the value of the character χ_A on principal ideals is given by:

$$\chi_A((\lambda)) = \left(\frac{\lambda, 2}{r_1}\right) = \begin{cases} 1, & \text{if} \quad \lambda \equiv \pm 1 \pmod{r_1^3}, \\ -1, & \text{if} \quad \lambda \equiv \pm 3 \pmod{r_1^3}. \end{cases} \tag{2.3}$$

Let (a, b) be a solution of $a^2+b^2 q = p^{h(-q)}$. As the integers $a+b\sqrt{-q}$ and $a-b\sqrt{-q}$ are coprime we may set

$$(a+b\sqrt{-q}) = p_1^{h(-q)}, \quad (a-b\sqrt{-q}) = p_2^{h(-q)}. \tag{2.4}$$

Now from (2.3) we first see, as $p \equiv \pm 1 \pmod 8$, that:

$$\chi_A(p_1)\chi_A(p_2) = \chi_A((p)) = 1, \tag{2.5}$$

so that from (2.3) and the fact that $h(-q)$ is odd:

$$\chi_A(p_1) = \chi_A(p_2) = \begin{cases} 1, & \text{if} \quad a+b\sqrt{-q} \equiv \pm 1 \pmod{r_1^3}, \\ -1, & \text{if} \quad a+b\sqrt{-q} \equiv \pm 3 \pmod{r_1^3}. \end{cases} \tag{2.6}$$

Let $\beta = 1$ or 3 be such that $q \equiv -\beta^2 \pmod{16}$. As $(\beta-\sqrt{-q})(\beta+\sqrt{-q}) \equiv 0 \pmod{r_1^4 r_2^4}$ and $(\beta-\sqrt{-q}, \beta+\sqrt{-q}) = 2$ there exists $\varepsilon = \pm 1$ such that $a+b\sqrt{-q} \equiv a+\varepsilon\beta b \pmod{r_1^3}$ and so

$$\chi_A(p_1) = \begin{cases} 1, & \text{if} \quad a+\varepsilon\beta b \equiv \pm 1 \pmod 8, \\ -1, & \text{if} \quad a+\varepsilon\beta b \equiv \pm 3 \pmod 8, \end{cases}$$

that is

$$\chi_A(p_1) = \chi_A(p_2) = \left(\frac{2}{a+\varepsilon\beta b}\right). \tag{2.7}$$

The integer a is odd or divisible by 4 according as $p \equiv 1$ or $-1 \pmod 8$ so that when $q \equiv -9 \pmod{16}$ we have

$$\left(\frac{2}{a+3b}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{a+b}\right)$$

which together with (2.7) proves

$$\chi_A(p_1) = \chi_A(p_2) = \left(\frac{-1}{p}\right)^{(q+1)/8}\left(\frac{2}{a+b}\right). \tag{2.8}$$

We next consider the extension B/k_r. By (2.1) of [4] we can suppose that q_1 ramifies in B and q_2 in B'. Then the character χ_B is given by

$$\chi_B((\lambda)) = \left[\frac{\lambda}{q_1}\right]_2 \times \operatorname{sgn} \lambda. \tag{2.9}$$

Let (e, f) be any solution of $p = e^2 - 2f^2$ where $e > 0$. Then we may set $p_1 = (e+f\sqrt{2})$, $p_2 = (e-f\sqrt{2})$, and we deduce from (2.9) that

$$\chi_B(p_1) = \chi_B(p_2) = \left[\frac{e+f\sqrt{2}}{q_1}\right]_2 = \left(\frac{e+f\sqrt{2}}{q}\right), \tag{2.10}$$

which together with (2.8) completes the proof of Theorem 3.

Remark. The class C_p of discriminant m is a fourth power or not according as $p^{h(m)/4}$ is represented by the principal class I or by the class J of order 2. Using the well-known representative of I and of J, and also the forms of discriminant $4m$ when m is odd, we obtain:

	C_p fourth power	C_p square, not fourth power
Theorem I	$p^{h(-r)/4} = X^2+rY^2$	$2p^{h(-r)/4} = X^2+rY^2$
Theorem II	$p^{h(-2r)/4} = X^2+2rY^2$	$p^{h(-2r)/4} = 2X^2+rY^2$
Theorem III	$p^{h(-2q)/4} = X^2+2qY^2$	$p^{h(-2q)/4} = 2X^2+qY^2$
Theorem IV	$\begin{cases} p^{h(-qr)/4} = X^2+XY+\frac{qr+1}{4}Y^2 \\ 4p^{h(-qr)/4} = X^2+qrY^2 \end{cases}$	$p^{h(-qr)/4} = qX^2+qXY+\frac{q+r}{4}Y^2$ $4p^{h(-qr)/4} = qX^2+rY^2$
Theorem V	$p^{h(2r)/4} = X^2-2rY^2$	$gp^{h(2r)/4} = X^2-2rY^2$
Theorem VI	$\begin{cases} p^{h(qr)/4} = X^2+XY+\frac{1-qr}{4}Y^2 \\ 4p^{h(qr)/4} = X^2-qrY^2 \end{cases}$	$gp^{h(qr)/4} = X^2+XY+\frac{1-qr}{4}Y^2$ $4gp^{h(qr)/4} = X^2-qrY^2$

In the cases (V), (VI) when $m>0$ the integer $g=-1$, q or r is such that the solvable non pellian equation is $X^2-qrY^2=g$.

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