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ON THE DIVISIBILITY OF THE CLASS NUMBER OF $Q(\sqrt{-pq})$ BY 16

by PHILIP A. LEONARD and KENNETH S. WILLIAMS*

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1. Introduction

Let $d(<0)$ denote a squarefree integer. The ideal class group of the imaginary quadratic field $Q(\sqrt{d})$ has a cyclic 2-Sylow subgroup of order $\geqq 8$ in precisely the following cases (see for example [**5**] and [**6**]):

(i) $d=-p,\ p=2g^2-h^2\equiv 1 \pmod 8,\ (g/p)=+1;$
(ii) $d=-2p,\ p=u^2-2v^2\equiv 1 \pmod 8$ with u chosen so that $u\equiv 1 \pmod 4),\ (u/p)=+1;$
(iii) $d=-2p,\ p\equiv 15 \pmod{16};$
(iv) $d=-pq,\ p\equiv 1 \pmod 4,\ q\equiv 3 \pmod 4,\ (q/p)=+1,\ (-q/p)_4=+1,$

where p and q denote primes and g, h, u and v are positive integers. The class number of $Q(\sqrt{d})$ is denoted by $h(d)$ and in the above cases $h(d)\equiv 0 \pmod 8$. For cases (i), (ii) and (iii) the authors [**6**] have given necessary and sufficient conditions for $h(d)$ to be divisible by 16. In this paper we do the same for case (iv) extending the results of Brown [**4**].

As the ideal class group of $Q(\sqrt{-pq})$ is isomorphic to the group (under composition) of classes of integral positive-definite binary quadratic forms $(a,b,c)=ax^2+bxy+cy^2$ of discriminant $b^2-4ac=-pq$, we can work with forms rather than ideals. In order to determine $h(-pq)$ modulo 16 we construct explicitly a form f of discriminant $-pq$ whose square is in the ambiguous class containing the form $(p,p,\frac{1}{4}(p+q))$ (see Theorem 1 in Section 2). The form f is given in terms of a solution in positive integers X, Y, Z of the Legendre equation

$$pX^2+qY^2-Z^2=0 \tag{1.1}$$

satisfying

$$(X,Y)=(Y,Z)=(Z,X)=1,\ p\nmid YZ,\ q\nmid XZ, \tag{1.2}$$

and

$$X \text{ odd},\ Y \text{ even},\ Z\equiv 1 \pmod 4). \tag{1.3}$$

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That there is a solution of (1.1) satisfying (1.2) follows immediately from Legendre's theorem in view of (iv). However we must justify that we can always find a solution with $Z \equiv 1 \pmod 4$. In order to see this we let $R+S\sqrt{q}$ be the fundamental unit (>1) of the real quadratic field $Q(\sqrt{q})$. As $q \equiv 3 \pmod 4$ we have

$$R^2 - qS^2 = +1.$$

It is well known that

$$R \equiv 2 \pmod 8,\ S \equiv 1 \pmod 2, \qquad \text{if} \quad q \equiv 3 \pmod 8,$$

$$R \equiv 0 \pmod 8,\ S \equiv 1 \pmod 2, \qquad \text{if} \quad q \equiv 7 \pmod 8,$$

and hence

$$R_1 = R^2 + qS^2 \equiv 7 \pmod 8,\ S_1 = 2RS \equiv 0 \pmod 4, \qquad R_1^2 - qS_1^2 = +1.$$

Hence if Z is even (so that X and Y are both odd) we can replace the solution (X, Y, Z) of (1.1) by the solution (X_1, Y_1, Z_1) given by

$$X_1 = X,\ Y_1 = RY + SZ,\ Z_1 = qSY + RZ,$$

for which Z_1 is odd. Further if $Z \equiv 3 \pmod 4$ (in which case X is odd and Y is even) we can replace the solution (X, Y, Z) by the solution (X_2, Y_2, Z_2) given by

$$X_2 = X,\ Y_2 = R_1 Y + S_1 Z,\ Z_2 = qS_1 Y + R_1 Z,$$

for which $Z_2 \equiv 1 \pmod 4$.

Our main result is the following theorem.

Theorem 2. *If p and q are primes such that*

$$p \equiv 1 \pmod 4,\ q \equiv 3 \pmod 4,\ \left(\frac{p}{q}\right) = +1,\ \left(\frac{-q}{p}\right)_4 = +1, \tag{1.4}$$

and (X, Y, Z) is any solution in positive integers of (1.1) *which satisfies* (1.2) *and* (1.3), *then*

$$h(-pq) \equiv 0 \pmod{16} \Leftrightarrow \left(\frac{Z}{p}\right)_4 = \left(\frac{2X}{Z}\right).$$

We remark that $(Z/p)_4$ is well-defined as $(Z/p) = +1$ and $p \equiv 1 \pmod 4$. To see that $(Z/p) = +1$ we perform the following calculation: letting $Y = 2^n Y_1$, Y_1 odd, we have, using

(1.1) and (1.2),

$$\begin{aligned}\left(\frac{Z}{p}\right)=\left(\frac{Z^2}{p}\right)_4=\left(\frac{qY^2}{p}\right)_4&=\left(\frac{q}{p}\right)_4\left(\frac{Y}{p}\right)=\left(\frac{q}{p}\right)_4\left(\frac{2}{p}\right)^n\left(\frac{Y_1}{p}\right)\\&=\left(\frac{q}{p}\right)_4\left(\frac{2}{p}\right)\left(\frac{p}{Y_1}\right)\quad(\text{as } n=1 \text{ when } p\equiv 5 (\text{mod } 8))\\&=\left(\frac{q}{p}\right)_4\left(\frac{-1}{p}\right)_4\left(\frac{pX^2}{Y_1}\right)\quad(\text{as } p\equiv 1(\text{mod } 4))\\&=\left(\frac{-q}{p}\right)_4\left(\frac{Z^2}{Y_1}\right)\\&=+1.\quad(\text{by } (1.4)).\end{aligned}$$

2. Square root of $(p,p,(p+q)/4)$

In this section we construct a form f of discriminant $-pq$ such that $f^2\sim(p,p,\frac{1}{4}(p+q))$.

As $(X,Y)=1$ there exists an integer u_0 such that $u_0X\equiv 1(\text{mod } Y)$. If the integer $e=(u_0X-1)/Y$ is odd we set $u=u_0$. If the integer $(u_0X-1)/Y$ is even then the integer

$$e=\frac{(u_0+Y)X-1}{Y}=\frac{u_0X-1}{Y}+X$$

is odd and we set $u=u_0+Y$. Thus the integers u and e satisfy

$$uX\equiv 1(\text{mod } Y),\ u \text{ odd},\ e=(uX-1)/Y \text{ odd}. \tag{2.1}$$

Next, appealing to (1.1) and (2.1), we have

$$X(pX-uZ^2)\equiv 0(\text{mod } Y)$$

so that, as $(X,Y)=1$, we have

$$pX-uZ^2\equiv 0(\text{mod } Y).$$

Hence we can define a positive integer a and an integer b by

$$a=Z,\ b=(pX-ua^2)/Y. \tag{2.2}$$

From (2.2) we obtain

$$pX-bY=ua^2. \tag{2.3}$$

Also using (1.1), (2.1) and (2.2) we get

$$bX+qY=-ea^2, \tag{2.4}$$

and

$$b^2+pq=(pe^2+qu^2)a^2. \tag{2.5}$$

From (1.4) and (2.1) we see that $pe^2+qu^2 \equiv 0 \pmod 4$ so we can define an integer c by

$$c=(pe^2+qu^2)/4. \tag{2.6}$$

Thus, from (2.5) and (2.6), we have

$$b^2-4a^2c=-pq, \tag{2.7}$$

showing that the form (a,b,ac) has discriminant $-pq$. We note that (2.7) shows that b is odd.

With a, b and c as defined in (2.2) and (2.6) we prove the following theorem.

Theorem 1. $(a,b,ac)^2 \sim (p,p,(p+q)/4)$.

Proof. We define integers v, α and β by

$$v=2Y, \quad \alpha=(u+e)/2, \quad \beta=X+Y. \tag{2.8}$$

Appealing to (1.1), (2.3) and (2.7) we obtain, on completing the square for u,

$$a^2u^2+buv+cv^2=p, \tag{2.9}$$

and appealing to (2.3), (2.4), (2.7) and (2.8), we obtain

$$\begin{aligned} bu+2cv &= \frac{1}{a^2}(bua^2+4a^2cY) \\ &= \frac{1}{a^2}(bua^2+(b^2+pq)Y) \\ &= \frac{1}{a^2}(b(bY+ua^2)+pqY) \\ &= \frac{1}{a^2}(bpX+pqY), \end{aligned}$$

that is

$$bu+2cv=-pe. \tag{2.10}$$

Hence from (2.3), (2.8) and (2.10) we have

$$\alpha=(pu-bu-2cv)/2p, \quad \beta=(2ua^2+bv+pv)/2p. \tag{2.11}$$

Thus from (2.9) and (2.11) we obtain

$$u\beta-v\alpha=1 \tag{2.12}$$

and

$$2a^2u\alpha+bu\beta+bv\alpha+2cv\beta=p. \tag{2.13}$$

Hence from (2.7), (2.9) (2.12) and (2.13) and the identity

$$(2a^2u\alpha+bu\beta+bv\alpha+2cv\beta)^2-4(a^2u^2+buv+cv^2)(a^2\alpha^2+b\alpha\beta+c\beta^2)=(u\beta-v\alpha)^2(b^2-4a^2c),$$

we deduce

$$a^2\alpha^2+b\alpha\beta+c\beta^2=(p+q)/4. \tag{2.14}$$

Hence the unimodular transformation with matrix$\left[\begin{smallmatrix} u & \alpha \\ v & \beta \end{smallmatrix}\right]$ changes the form (a^2, b, c) into

$$(a^2u^2+buv+cv^2,\ 2a^2u\alpha+bu\beta+bv\alpha+2cv\beta,\ a^2\alpha^2+b\alpha\beta+c\beta^2)=(p, p, (p+q)/4).$$

Thus we have (see for example [**3**, p. 185])

$$(a, b, ac)^2 \sim (a^2, b, c) \sim (p, p, (p+q)/4),$$

which completes the proof of Theorem 1.

3. Determination of $h(-pq)$modulo 16; Proof of Theorem 2

By Theorem 1 the class of the form (a, b, ac) is of order 4 and so as the 2-Sylow subgroup of the class group of forms of discriminant $-pq$ is cyclic, the form (a, b, ac) is equivalent to the square of a form (r, s, t), where we may take $(r, 2pqac)=1$. Hence (a, b, ac) represents r^2 primitively so that there are integers x and y such that

$$r^2=ax^2+bxy+acy^2, \quad x>0, \quad (x, y)=1. \tag{3.1}$$

We define non-negative integers S and T by

$$S=|2Xx-aey|, \qquad T=|2Yx-auy|. \tag{3.2}$$

Appealing to (1.1), (2.1), (2.2), (2.6) and (3.1) we obtain

$$4ar^2 = pS^2 + qT^2. \tag{3.3}$$

From (3.3) we easily deduce that S and T are positive.

We now show that S and T have no odd common divisors greater than 1. Suppose k is an odd prime divisor of both S and T. Then k divides

$$\begin{aligned} &u(2Xx - aey) - e(2Yx - auy) \\ &= 2x(uX - eY) \\ &= 2x \qquad \text{(by (2.1))}, \end{aligned}$$

that is $k|x$. Further from (3.3) we have $k|ar^2$ so that $k|a$ or $k|r$. If $k|a$ from (3.1) we have $k|r$ contradicting $(r, a) = 1$. If $k|r$ by (3.1) we have $k|acy^2$ contradicting $(r, ac) = (x, y) = 1$.

Similarly we can show that T and apr have no odd common divisors greater than 1.

We note that as a is represented by (a, b, ac) and the class of the form (a, b, ac) is in the principal genus we have

$$\left(\frac{a}{p}\right) = +1. \tag{3.4}$$

Further by (1.3) and (2.2) we have

$$a \equiv 1 \pmod 4. \tag{3.5}$$

Then

$$\begin{aligned} \left(\frac{r}{p}\right)\left(\frac{a}{p}\right)_4 &= \left(\frac{ar^2}{p}\right)_4 = \left(\frac{2}{p}\right)\left(\frac{4ar^2}{p}\right)_4 \\ &= \left(\frac{-1}{p}\right)_4\left(\frac{qT^2}{p}\right)_4 \qquad \text{(by (3.3))} \\ &= \left(\frac{-q}{p}\right)_4\left(\frac{T}{p}\right), \end{aligned}$$

that is (by (1.4))

$$\left(\frac{r}{p}\right)\left(\frac{a}{p}\right)_4 = \left(\frac{T}{p}\right) = \left(\frac{2}{p}\right)^n\left(\frac{t}{p}\right), \tag{3.6}$$

where

$$T = 2^n t, \qquad t \text{ odd}. \tag{3.7}$$

Then

$$\begin{aligned}
\left(\frac{t}{p}\right) &= \left(\frac{p}{t}\right) \\
&= \left(\frac{pS^2}{t}\right) \\
&= \left(\frac{4ar^2}{t}\right) && \text{(by (3.3))} \\
&= \left(\frac{a}{t}\right) \\
&= \left(\frac{t}{a}\right) && \text{(by (3.5))} \\
&= \left(\frac{2}{a}\right)^n \left(\frac{T}{a}\right) && \text{(by (3.7))} \\
&= \left(\frac{2}{a}\right)^n \left(\frac{|2Yx - auy|}{a}\right) \\
&= \left(\frac{2}{a}\right)^n \left(\frac{2Yx - auy}{a}\right) && \text{(by (3.5))} \\
&= \left(\frac{2}{a}\right)^{n+1} \left(\frac{Y}{a}\right)\left(\frac{x}{a}\right) \\
&= \left(\frac{2}{a}\right)^{n+1} \left(\frac{Y}{a}\right)\left(\frac{b}{a}\right)\left(\frac{y}{a}\right) && \text{(by (3.1)).}
\end{aligned}$$

Now set

$$|y| = 2^m y_1, \quad y_1 \text{ odd}, \quad y_1 > 0,$$

so appealing to (3.1) and (3.5) we have

$$\left(\frac{y}{a}\right) = \left(\frac{|y|}{a}\right) = \left(\frac{2}{a}\right)^m \left(\frac{y_1}{a}\right) = \left(\frac{2}{a}\right)^m \left(\frac{a}{y_1}\right) = \left(\frac{2}{a}\right)^m,$$

giving

$$\left(\frac{t}{p}\right) = \left(\frac{2}{a}\right)^{m+n+1} \left(\frac{bY}{a}\right).$$

Next as $bY = pX - ua^2$ and using (3.4) we have

$$\left(\frac{bY}{a}\right) = \left(\frac{pX}{a}\right) = \left(\frac{a}{p}\right)\left(\frac{X}{Z}\right) = \left(\frac{X}{Z}\right),$$

so

$$\left(\frac{t}{p}\right) = \left(\frac{2}{Z}\right)^{m+n+1}\left(\frac{X}{Z}\right),$$

giving

$$\left(\frac{r}{p}\right) = \left(\frac{2}{p}\right)^{n}\left(\frac{2}{Z}\right)^{m+n+1}\left(\frac{X}{Z}\right)\left(\frac{a}{p}\right)_4. \tag{3.8}$$

Taking (1.1) modulo 8 we obtain $p + qY^2 \equiv 1 \pmod 8$, so that

$$p \equiv 1 \pmod 8 \Rightarrow Y \equiv 0 \pmod 4,$$

$$p \equiv 5 \pmod 8 \Rightarrow Y \equiv 2 \pmod 4.$$

We now treat the case $p \equiv 1 \pmod 8$: we have

$$m = 0 \Rightarrow y \text{ odd} \Rightarrow T \text{ odd} \Rightarrow n = 0;$$

$$m = 1 \Rightarrow 2 \| y \Rightarrow 2 \| T \Rightarrow n = 1;$$

$$m = 2 \Rightarrow 4 \| y \Rightarrow 4 \| T \Rightarrow n = 2;$$

$$m \geqq 3 \Rightarrow 8 | y \Rightarrow x \text{ odd} \Rightarrow a \equiv 1 \pmod 8 \Rightarrow \left(\frac{2}{Z}\right) = +1;$$

so that in each case

$$\left(\frac{2}{p}\right)^{n}\left(\frac{2}{Z}\right)^{m+n} = 1.$$

For the case $p \equiv 5 \pmod 8$ we have

$$m = 0 \Rightarrow y \text{ odd} \Rightarrow T \text{ odd} \Rightarrow n = 0;$$

$$m = 1 \Rightarrow 2 \| y \Rightarrow 4 | S, 2 \| T \Rightarrow pS^2 + qT^2 \equiv 12 \pmod{16}$$

$$\Rightarrow ar^2 \equiv 3 \pmod 4, \text{ which is impossible;}$$

$$m = 2 \Rightarrow x \text{ odd, } 4 \| y \Rightarrow a \equiv 5 \pmod 8 \Rightarrow \left(\frac{2}{Z}\right) = -1;$$

$$m \geqq 3 \Rightarrow x \text{ odd}, \ 8|y \Rightarrow \begin{cases} a \equiv 1 \pmod 8 & \Rightarrow \left(\frac{2}{Z}\right) = +1, \\ 4\|T & \Rightarrow n = 2; \end{cases}$$

so that again in each case we have

$$\left(\frac{2}{p}\right)^n \left(\frac{2}{Z}\right)^{m+n} = 1.$$

Hence by (3.8) we have

$$\left(\frac{r}{p}\right) = \left(\frac{2}{Z}\right)\left(\frac{X}{Z}\right)\left(\frac{Z}{p}\right)_4.$$

Now by a theorem of Bauer [**1**] (see also [**2**, Theorem 6])

$$h(-pq) \equiv 0 \pmod{16} \Leftrightarrow \left(\frac{r}{p}\right) = +1$$

so we have

$$h(-pq) \equiv 0 \pmod{16} \Leftrightarrow \left(\frac{Z}{p}\right)_4 = \left(\frac{2X}{Z}\right).$$

This completes the proof of Theorem 2.

We remark that Theorem 2 of Brown [**4**] is the special case of our Theorem 2 which arises when (1.1) has a solution with $X = 1$.

4. Examples

Example 1. $p = 5$, $q = 19$.
Here

$$\left(\frac{q}{p}\right) = \left(\frac{19}{5}\right) = 1, \qquad \left(\frac{-q}{p}\right)_4 = \left(\frac{-19}{5}\right)_4 = +1.$$

A solution of (1.1)–(1.3) is given by

$$X = 1, \quad Y = 2, \quad Z = 9$$

so

$$\left(\frac{Z}{p}\right)_4 = \left(\frac{9}{5}\right)_4 = \left(\frac{3}{5}\right) = -1, \quad \left(\frac{2X}{Z}\right) = \left(\frac{2}{9}\right) = +1,$$

and Theorem 2 implies $h(-pq)=h(-95)\equiv 8 \pmod{16}$. Indeed $h(-95)=8$.

Example 2. $p=37$, $q=11$.
Here

$$\left(\frac{q}{p}\right)=\left(\frac{11}{37}\right)=\left(\frac{37}{11}\right)=\left(\frac{4}{11}\right)=+1, \quad \left(\frac{-q}{p}\right)_4=\left(\frac{-11}{37}\right)_4=\left(\frac{100}{37}\right)_4=\left(\frac{10}{37}\right)=+1.$$

We start with a solution of (1.1) and (1.2) for which Z is even, say,

$$X=1, \quad Y=7, \quad Z=24,$$

in order to illustrate how to obtain a solution which satisfies (1.3) as well. Since the fundamental unit of $Q(\sqrt{11})$ is $10+3\sqrt{11}$ we have

$$R=10, \quad S=3, \quad R_1=199, \quad S_1=60.$$

First we transform the solution (X, Y, Z) into a solution (X_1, Y_1, Z_1) with Z_1 odd:

$$X_1=X=1, \quad Y_1=RY+SZ=142, \quad Z_1=qSY+RZ=471.$$

As $Z_1\equiv 3 \pmod 4$ we transform the solution (X_1, Y_1, Z_1) into a solution (X_2, Y_2, Z_2) with $Z_2\equiv 1 \pmod 4$:

$$X_2=X_1=1, \quad Y_2=R_1Y_1+S_1Z_1=56518,$$
$$Z_2=qS_1Y_1+R_1Z_1=187449,$$

so that

$$\left(\frac{Z_2}{p}\right)_4=\left(\frac{187449}{37}\right)_4=\left(\frac{7}{37}\right)_4=\left(\frac{81}{37}\right)_4=+1, \quad \left(\frac{2X_2}{Z_2}\right)=\left(\frac{2}{187449}\right)=+1,$$

and Theorem 2 implies $h(-pq)=h(-407)\equiv 0 \pmod{16}$. Indeed $h(-407)=16$.

Example 3. $p=5$, $q=79$.
Here

$$\left(\frac{q}{p}\right)=\left(\frac{79}{5}\right)=+1, \quad \left(\frac{-q}{p}\right)_4=\left(\frac{-79}{5}\right)_4=+1.$$

A solution of (1.1) and (1.2) is given by

$$X=3, \quad Y=2, \quad Z=19.$$

As $Z \equiv 3 \pmod 4$ we transform this solution into one for which $Z \equiv 1 \pmod 4$ obtaining

$$X = 3, \quad Y = 52958, \quad Z = 470701,$$

so that

$$\left(\frac{Z}{p}\right)_4 = +1, \quad \left(\frac{2X}{Z}\right) = \left(\frac{2}{Z}\right)\left(\frac{3}{Z}\right) = (-1)(+1) = -1,$$

and Theorem 2 implies $h(-pq) = h(-395) \equiv 8 \pmod{16}$. Indeed $h(-395) = 8$.

This example illustrates Theorem 2 in a situation where (1.1) has no solution with $X = 1$ as

$$u^2 - 79v^2 = 5$$

is insolvable in integers u and v (see for example [7, Theorem 109]).

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Department of Mathematics
Arizona State University
Tempe, Arizona 85287, U.S.A.

Department of Mathematics
and Statistics
Carleton University
Ottawa, Ontario, Canada K1S 5B6