Ontario Mathematics Gazette 15 (1976), 42-44.

PEDOE'S FORMULATION OF URQUHART'S THEOREM

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Urquhart's theorem of Euclidean geometry states:

Let ℓ and ℓ' be two straight lines which intersect at A. Let B and C be points on ℓ with C between A and B. Let D and E be points on ℓ' with E between A and D. Suppose that BE and CD intersect at F. If AC + CF = AE + EF, then we have AB + BF = AD + DF.

In a recent article Pedoe [2] asserts that an equivalent version of Urquhart's theorem is the following:

If C and E are points on an ellipse with foci A and F, then $B = AC \cap EF$ and $D = AE \cap CF$ lie on a confocal ellipse.

Unfortunately, this is not quite correct as it stands. To make it correct, we must insert the requirement that C and E lie on opposite sides of AF. More precisely we have:

Let C and E be points on an ellipse ξ with foci A and F, and set $B = AC \cap EF$ and $D = AE \cap CF$. If C and E lie on opposite sides of the major axis of ξ, then B and D lie on a confocal ellipse; whereas, if C and E lie on the same side of the major axis, then B and D lie on a confocal hyperbola.

To prove this, we choose our co-ordinate system so that ξ has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < b < a$. Writing e for the eccentricity of ξ, we have $A = (-ae, 0)$, $F = (ae, 0)$. Since C and E lie on ξ, we can suppose $C = (a\cos\theta, b\sin\theta)$, $E = (a\cos\phi, b\sin\phi)$. Setting $\alpha = \frac{1}{2}(\phi + \theta)$, $\beta = \frac{1}{2}(\phi - \theta)$, a straightforward calculation shows that

$$B = \left[ae\,\frac{\cos\alpha(\sin\alpha + e\sin\beta)}{\cos\beta(\sin\beta + e\sin\alpha)},\ be\,\frac{(\sin^2\alpha - \sin^2\alpha)}{\cos\beta(\sin\beta + e\sin\alpha)}\right]$$

and

$$D = \left[ae\,\frac{\cos\alpha(-\sin\alpha + e\sin\beta)}{\cos\beta(\sin\beta - e\sin\alpha)},\ -be\,\frac{(\sin^2\alpha - \sin^2\beta)}{\cos\beta(\sin\beta - e\sin\alpha)}\right].$$

It is easily verified (remembering that $b^2 = a^2(1 - e^2)$) that B and D lie on the conic

$$(1) \qquad \frac{x^2}{L} + \frac{y^2}{M} = 1 ,$$

where

$$L = a^2e^2 \frac{\cos^2\alpha}{\cos^2\beta}, \quad M = a^2e^2 \frac{(\cos^2\alpha - \cos^2\beta)}{\cos^2\beta}.$$

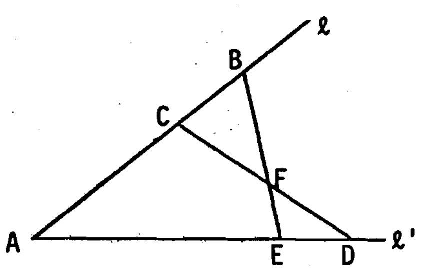
Since $L = a^2 + \lambda$, $M = b^2 + \lambda$, with

$$\lambda = a^2 \left(\frac{e^2\cos^2\alpha}{\cos^2\beta} - 1\right).$$

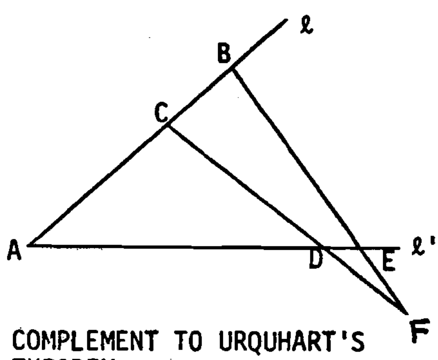
(1) is a conic confocal with ξ. Now, if C and E lie on opposite sides of AF, the chord joining them, viz., $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\beta$, meets the x-axis between $x = -a$ and $x = a$, so we have $-1 \le \frac{\cos\beta}{\cos\alpha} \le 1$, that is $\cos^2\alpha - \cos^2\beta \ge 0$, i.e., $M \ge 0$ and (1) is an ellipse (possibly degenerate). On the other hand, if C and E lie on the same side of the major axis, then reasoning as above, we obtain $M \le 0$, so that (1) is a hyperbola. This completes the proof.

The case when C and E lie on the same side of AF leads to the following complement to Urquhart's theorem (see for example [1]):

Let ℓ and ℓ' be two straight lines which intersect at A. Let B and C be points on ℓ with C between A and B. Let D and E be points on ℓ' with D between A and E. Suppose that BE and CD intersect at F. If AC + CF = AE + EF, then we have AB - BF = AD - DF.



URQUHART'S THEOREM:
AC+CF = AE+EF => AB+BF = AD+DF



COMPLEMENT TO URQUHART'S THEOREM:
AC+CF = AE+EF => AB-BF = AD-df

References

1. H. Grossman, Urquhart's quadrilateral theorem, The Mathematics Teacher, 66 (1973), 643-644.

2. D. Pedoe, The most elementary theorem of Euclidean geometry, Mathematics Magazine 49 (1976), 40-42.