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Mertens' Theorem for Arithmetic Progressions

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Let k be an integer $\geqslant 1$ and let l be an integer such that $1 \leqslant l \leqslant k$, $(l, k) = 1$. An asymptotic formula (valid for large x) is obtained for the product

$$\prod_{p \leqslant x, p \equiv l (\mathrm{mod}\, k)} \left(1 - \frac{1}{p}\right),$$

generalizing a familiar result of Mertens.

1. INTRODUCTION

Let k be an integer $\geqslant 1$ and let l be an integer such that $1 \leqslant l \leqslant k$, $(l, k) = 1$. In this paper we obtain an asymptotic formula for

$$\prod_{p \leqslant x, p \equiv l (\mathrm{mod}\, k)} \left(1 - \frac{1}{p}\right),$$

as $x \to \infty$, where the product is taken over primes p in the specified arithmetic progression. Our formula (Section 3, Theorem 1) generalizes the familiar result of Mertens,

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = e^{-c} (\log x)^{-1} + O((\log x)^{-2}), \tag{1.1}$$

where c denotes Euler's constant, and also the recent result of Uchiyama [5],

$$\prod_{p \leqslant x, p \equiv l (\mathrm{mod}\, 4)} \left(1 - \frac{1}{p}\right) = \left(\beta_l e^{-c} \prod_{p \equiv l (\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)\right)^{1/2} (\log x)^{-1/2} + O((\log x)^{-3/2}), \tag{1.2}$$

where $l = 1, 3$ and $\beta_1 = \pi$, $\beta_3 = \frac{1}{2}\pi$.

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2. Dirichlet Series $K(s, \chi)$

For each character χ (modulo k) we define a completely multiplicative function $k_\chi(n)$ ($n = 1, 2, \ldots$) by setting for primes p

$$k_\chi(p) = p\left\{1 - \left(1 - \frac{\chi(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-\chi(p)}\right\}. \tag{2.1}$$

Now

$$\begin{aligned}
k_\chi(p) &= \sum_{n=2}^{\infty} \frac{1}{p^{n-1}}\left\{\chi(p)\,\frac{\chi(p)(\chi(p)+1)\cdots(\chi(p)+n-2)}{(n-1)!}\right.\\
&\qquad \left. - \frac{\chi(p)(\chi(p)+1)\cdots(\chi(p)+n-1)}{n!}\right\}\\
&= \sum_{n=2}^{\infty} \frac{1}{p^{n-1}}\,\frac{\chi(p)(\chi(p)+1)\cdots(\chi(p)+n-2)(n-1)(\chi(p)-1)}{n!}\\
&= \frac{1}{p}\,\frac{\chi(p)(\chi(p)-1)}{2} + \chi(p)(\chi(p)-1)\\
&\qquad \times \sum_{n=3}^{\infty} \frac{(\chi(p)+1)\cdots(\chi(p)+n-2)}{p^{n-1}(n-1)!}\,\frac{n-1}{n}.
\end{aligned}$$

In the last sum we use the fact that $|\chi(k)+j| \leqslant j+1$ for $j = 2, 3, \ldots$ and $|(\chi(p)-1)(\chi(p)+1)| = |\chi(p)^2 - 1| \leqslant 2$. Thus

$$\begin{aligned}
|k_\chi(p)| &\leqslant \frac{1}{p} + \sum_{n=3}^{\infty} \frac{1}{p^{n-1}}\,\frac{n-1}{n}\\
&\leqslant \frac{1}{p} + \sum_{n=3}^{\infty} \frac{1}{p^{n-1}} = \frac{1}{p-1},
\end{aligned}$$

and so for $s = \sigma + it$ we have

$$\left|\frac{k_\chi(p)}{p^s}\right| \leqslant \frac{1}{(p-1)\,p^\sigma} \qquad (< 1 \text{ for } \sigma > 0), \tag{2.2}$$

so that $\sum_p \{\sum_{n=1}^{\infty} (k_\chi(p)/p^s)^n\}$ converges absolutely for $\sigma > 0$, as

$$\begin{aligned}
\sum_p \left|\sum_{n=1}^{\infty} \left(\frac{k_\chi(p)}{p^s}\right)^n\right| &\leqslant \sum_p \sum_{n=1}^{\infty} \left(\frac{1}{(p-1)\,p^\sigma}\right)^n = \sum_p \frac{1}{(p-1)p^\sigma - 1}\\
&\leqslant \frac{2^{\sigma+1}}{2^\sigma - 1} \sum_p \frac{1}{p^{\sigma+1}} < +\infty, \qquad \text{for} \quad \sigma > 0.
\end{aligned}$$

Hence

$$\prod_p \left(1 - \frac{k_\chi(p)}{p^s}\right)^{-1} = \prod_p \left(1 + \sum_{n=1}^{\infty} \left(\frac{k_\chi(p)}{p^s}\right)^n\right)$$

converges absolutely for $\sigma > 0$. Thus the Dirichlet series

$$K(s, \chi) = \sum_{n=1}^{\infty} \frac{k_\chi(n)}{n^s}$$

converges absolutely for $\sigma > 0$, and $K(s, \chi) = \prod_p (1 - (k_\chi(p)/p^s))^{-1}$ for $\sigma > 0$. In particular we have

$$K(1, \chi) = \sum_{n=1}^{\infty} \frac{k_\chi(n)}{n} = \prod_p \left(1 - \frac{k_\chi(p)}{p}\right)^{-1} \neq 0. \tag{2.3}$$

Moreover from (2.2) we have

$$\sum_{p \geqslant x} \frac{|k_\chi(p)|}{p} = O\left(\frac{1}{x}\right),$$

and a standard argument shows that

$$\prod_{p \leqslant x} \left(1 - \frac{k_\chi(p)}{p}\right)^{-1} = K(1, \chi) + O\left(\frac{1}{x}\right). \tag{2.4}$$

The Dirichlet L-series corresponding to χ is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{2.5}$$

It is well-known that for $\chi \neq \chi_0$ (the principal character mod k) the series in (2.5) converges for $\sigma > 0$ and that

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1} \neq 0$$

and (see for example [3, Section 109])

$$\prod_{p \leqslant x} \left(1 - \frac{\chi(p)}{p}\right) = \frac{1}{L(1, \chi)} + O\left(\frac{1}{\log x}\right), \quad \text{as} \quad x \to \infty. \tag{2.6}$$

3. Asymptotic Formula

We prove

Theorem 1.

$$\prod_{p\leqslant x,\, p\equiv l(k)}\left(1-\frac{1}{p}\right)=\left(e^{-c}\frac{k}{\phi(k)}\prod_{\chi\neq\chi_0}\left(\frac{K(1,\chi)}{L(1,\chi)}\right)^{\bar{\chi}(l)}\right)^{1/\phi(k)}(\log x)^{-1/\phi(k)}$$
$$+\,O((\log x)^{-1/\phi(k)-1}),$$

where $\phi(k)$ is Euler's totient function, the product on the right-hand side is taken over all characters χ (modulo k) different from the principal character χ_0 (modulo k), and the constant implied by the O-symbol depends only on k.

Proof. As

$$\sum_{\chi}\chi(p)\,\bar{\chi}(l)=\begin{cases}\phi(k), & \text{if } p\equiv l \pmod{k},\\ 0, & \text{otherwise,}\end{cases}$$

we have

$$\prod_{p\leqslant x,\, p\equiv l(k)}\left(1-\frac{1}{p}\right)^{\phi(k)}=\prod_{\chi}\left\{\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\chi(p)}\right\}^{\bar{\chi}(l)}. \tag{3.1}$$

Now for $\chi\neq\chi_0$ (using (2.1), (2.4), (2.6)) we have

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\chi(p)}=\prod_{p\leqslant x}\left(1-\frac{\chi(p)}{p}\right)\prod_{p\leqslant x}\left(1-\frac{k_\chi(p)}{p}\right)^{-1}$$
$$=\left\{\frac{1}{L(1,\chi)}+O\left(\frac{1}{\log x}\right)\right\}\left\{K(1,\chi)+O\left(\frac{1}{x}\right)\right\}.$$

Hence for $\chi\neq\chi_0$,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\chi(p)}=\frac{K(1,\chi)}{L(1,\chi)}+O\left(\frac{1}{\log x}\right). \tag{3.2}$$

Further, from (1.1) we have

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\chi_0(p)}=e^{-c}\frac{k}{\phi(k)}(\log x)^{-1}+O((\log x)^{-2}). \tag{3.3}$$

The theorem now follows from (3.1), (3.2) and (3.3).

We remark that the error term in the theorem can be improved if we make use of the prime number theorem to improve the error term in (1.1) and the prime number theorem for arithmetic progressions to improve the error term in (2.6).

4. Example

If χ is a real character (mod k) then from (2.1) we have

$$k_\chi(p) = \begin{cases} 0, & \text{if } \chi(p) = 1 \text{ or } 0, \\ \dfrac{1}{p}, & \text{if } \chi(p) = -1, \end{cases}$$

and so from (2.3) we deduce that

$$K(1, \chi) = \prod_{\chi(p)=-1} \left(1 - \frac{1}{p^2}\right)^{-1}. \tag{4.1}$$

All characters (mod k) are real if and only if k is a divisor of 24. Taking $k = 24$ (the other cases can be derived from this) we have

THEOREM 2. *For* $l \equiv 1, 5, 7, 11, 13, 17, 19, 23$ *we have*

$$\prod_{p \leqslant x,\, p \equiv l(24)} \left(1 - \frac{1}{p}\right) = (\alpha_l e^{-c})^{1/8} \left(\prod_{p \equiv l(24)} \left(1 - \frac{1}{p^2}\right)\right)^{1/2} (\log x)^{-1/8} + O((\log x)^{-9/8}),$$

where

$$\alpha_1 = \frac{2\pi^4}{9c_1c_2c_3}, \qquad \alpha_5 = \frac{2c_1c_2}{c_3}, \qquad \alpha_7 = \frac{c_2c_3}{2c_1}, \qquad \alpha_{11} = \frac{2c_1c_3}{c_2},$$

$$\alpha_{13} = \frac{2c_1c_3}{c_2}, \qquad \alpha_{17} = \frac{9c_2c_3}{8c_1}, \qquad \alpha_{19} = \frac{8c_1c_2}{c_3}, \qquad \alpha_{23} = \frac{\pi^4}{8c_1c_2c_3},$$

$$c_1 = \log(1 + \sqrt{2}), \qquad c_2 = \log(2 + \sqrt{3}), \qquad c_3 = \log(5 + 2\sqrt{6}).$$

Proof. The $\phi(24) - 1 = 7$ non-principal characters (mod 24) are given by

$$\chi_1(n) = \begin{cases} +1, & n \equiv 1, 7, 13, 19\ (24), \\ -1, & n \equiv 5, 11, 17, 23\ (24), \end{cases}$$

$$\chi_2(n) = \begin{cases} +1, & n \equiv 1, 5, 13, 17\ (24), \\ -1, & n \equiv 7, 11, 19, 23\ (24), \end{cases}$$

$$\chi_3(n) = \begin{cases} +1, & n \equiv 1, 5, 7, 11\ (24), \\ -1, & n \equiv 13, 17, 19, 23\ (24), \end{cases}$$

$$\chi_4(n) = \begin{cases} +1, & n \equiv 1, 11, 13, 23\ (24), \\ -1, & n \equiv 5, 7, 17, 19\ (24), \end{cases}$$

$$\chi_5(n) = \begin{cases} +1, & n \equiv 1, 7, 17, 23\ (24), \\ -1, & n \equiv 5, 11, 13, 19\ (24), \end{cases}$$

$$\chi_6(n) = \begin{cases} +1, & n \equiv 1, 5, 19, 23\ (24), \\ -1, & n \equiv 7, 11, 13, 17\ (24), \end{cases}$$

$$\chi_7(n) = \begin{cases} +1, & n \equiv 1, 11, 17, 19\ (24), \\ -1, & n \equiv 5, 7, 13, 23\ (24), \end{cases}$$

and the values of $K(1, \chi_i)$ $(1 \leqslant i \leqslant 7)$ follow immediately from (4.1). From [2, Theorem 217] we have

$$L(1, \chi_1) = \pi/(2\sqrt{3}), \qquad L(1, \chi_2) = \pi/3, \qquad L(1, \chi_3) = \pi/\sqrt{6},$$

$$L(1, \chi_4) = c_2/\sqrt{3}, \qquad L(1, \chi_5) = 2\sqrt{2}\, c_1/3, \qquad L(1, \chi_6) = c_3/\sqrt{6},$$

$$L(1, \chi_7) = \pi/(3\sqrt{2}).$$

Putting these values in Theorem 1 gives Theorem 2.

5. Application

Rieger [4] has recently proved that if T is a set of primes such that

$$\sum_{p \leqslant x,\, p \in T} \frac{\log p}{p} \sim \tau \log x, \qquad \text{as} \quad x \to +\infty,$$

where $\tau \equiv \tau(T) > 0$, then

$$\sum_{\substack{m \leqslant x \\ p \mid m \Rightarrow p \in T}} \frac{1}{m} \sim \frac{e^{-c\tau}}{\Gamma(\tau + 1)} \prod_{p \leqslant x,\, p \in T} \left(1 - \frac{1}{p}\right)^{-1}, \qquad \text{as} \quad x \to +\infty.$$

Taking T to be the set of primes $\equiv l \pmod{k}$, so that $\tau = 1/\phi(k)$, and appealing to Theorem 1 we have

THEOREM 3.

$$\sum_{\substack{m \leqslant x \\ p \mid m \Rightarrow p \equiv l \pmod{k}}} \frac{1}{m} \sim \frac{1}{\Gamma\left(1 + \dfrac{1}{\phi(k)}\right)} \left(\frac{\phi(k)}{k} \prod_{\chi \neq \chi_0} \left(\frac{L(1, \chi)}{K(1, \chi)}\right)^{\bar{\chi}(l)}\right)^{1/\phi(k)} \times (\log x)^{1/\phi(k)},$$

as $x \to \infty$.

Professor Paul T. Bateman has indicated to the author that an alternative proof of Theorem 3 can be given by using results given in Sections 181 and 183 of [3].

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