$(\tau\sqrt{5}, \tau, \tau^{-2})$, $(2\tau, 2, 2\tau^{-1})$, $(\tau^2, \tau^2, \tau^{-1}\sqrt{5})$, $(\sqrt{5}, \sqrt{5}, \sqrt{5})$. Symmetry is that of the cube.

6c. Edge-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}$, with $\alpha = 3^{-1/2}\tau^{-3}$: $(\tau^2, \tau^{-2}, \alpha\tau^4)$, $(\tau^2, \tau^{-1}, \alpha\tau^2\sqrt{5})$, $(\tau^2, 1, \alpha)$, $(\sqrt{5}, \tau^{-1}, \alpha\tau^5)$, $(\sqrt{5}, 1, \alpha\tau^3\sqrt{5})$, $(\sqrt{5}, \tau, \alpha\tau)$, $(2, 0, 2\alpha\tau^4)$, $(2, 1, 3\alpha\tau^3)$, $(2, \tau, \alpha\tau^4)$, $(2, 2, 0)$, $(\tau, 0, 4\alpha\tau^3)$, $(\tau, \tau^{-1}, \alpha\tau^2(2\tau^2+1))$, $(\tau, \tau, \alpha(\tau^5+1))$, $(\tau, 2, 2\alpha\tau^3)$, $(\tau, \sqrt{5}, \alpha\tau^3)$, $(1, 0, \alpha(\tau^6+1))$, $(1, 1, \alpha\tau^6)$, $(1, \tau, \alpha\tau^4\sqrt{5})$, $(1, \sqrt{5}, \alpha\tau^3\sqrt{5})$, $(1, \tau^2, \alpha\tau^2)$, $(\tau^{-1}, \tau^{-1}, \alpha\tau(\tau^5+1))$, $(\tau^{-1}, 2, 2\alpha\tau^4)$, $(\tau^{-1}, \tau^2, \alpha\tau^2\sqrt{5})$, $(0, \tau^{-2}, 3\alpha\tau^4)$, $(0, \sqrt{5}, 3\alpha\tau^3)$, $(0, \tau^2, 3\alpha\tau^2)$. Symmetry is that of the hexagonal prism.

6d. Face-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}$, with $\beta = 5^{-1/4}\tau^{-5/2}$: $(\tau^2, 0, 2\beta\tau^2)$ $(\tau^2, \tau^{-1}, \beta\tau^3)$, $(\tau^2, 1, \beta\tau)$, $(\sqrt{5}, 0, 2\beta\tau^3)$, $(\sqrt{5}, 1, \beta\tau^4)$, $(\sqrt{5}, \tau, \beta\tau^2)$, $(2, \tau^{-1}, \beta\tau^3\sqrt{5})$, $(2, \tau, \beta\tau^2\sqrt{5})$, $(2, 2, 0)$, $(\tau, \tau^{-2}, \beta\tau^5)$, $(\tau, 1, \beta\tau^2(3\tau-1))$, $(\tau, \tau, 3\beta\tau^2)$, $(\tau, 2, 2\beta\tau^2)$, $(\tau, \sqrt{5}, \beta\tau)$, $(1, \tau^{-1}, 3\beta\tau^3)$, $(1, 1, \beta\tau^2(\tau+3))$, $(1, 2, 2\beta\tau^3)$, $(1, \sqrt{5}, \beta\tau^4)$, $(1, \tau^2, \beta)$, $(\tau^{-1}, 0, 2\beta\tau^4)$, $(\tau^{-1}, \tau, \beta\tau^5)$, $(\tau^{-1}, \tau^2, \beta\tau^3)$. Symmetry is that of the decagonal prism.

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ON $\sum_{n=1}^{\infty}(1/n^{2k})$

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In this note we give a simple proof of the well-known result ([1], [3])

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{2^{2k-1}\pi^{2k}B_k}{(2k)!}, \qquad k = 1, 2, 3, \cdots,$$

where B_k is the kth Bernoulli number, defined by

$$\sum_{k=1}^{\infty} B_k \frac{x^{2k}}{2k!} = 1 - \frac{x}{2}\cot\frac{x}{2}, \qquad |x| < 2\pi.$$

The proof is accomplished by estimating the sum $\sum_{r=1}^{n}\cot^{2k}(r\pi/2n+1)$, for large n, in two different ways (Lemmas 1 and 2).

LEMMA 1.

$$\lim_{n\to\infty}\frac{1}{(2n)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right) = \frac{2^{2k-1}}{(2k)!}B_k, \qquad k = 1, 2, 3, \cdots.$$

Proof. For $k = 1, 2, 3, \cdots$, let

$$s_n(k) = \frac{1}{(2n)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right).$$

Now the numbers $\cot(r\pi/2n+1)$, $r = \pm 1, \pm 2, \cdots, \pm n$, are the $2n$ roots of $(z+i)^{2n+1} - (z-i)^{2n+1} = 0$. This equation can be written

$$(1)\qquad \binom{2n+1}{1}z^{2n} - \binom{2n+1}{3}z^{2n-2} + \cdots + (-1)^n\binom{2n+1}{2n+1} = 0.$$

We note that $2(2n)^{2k}s_n(k)$ is the sum of the $2k$th powers of the roots of (1). Thus, by Newton's identity [2] for $n \geqq k$, we have on dividing through by $2(2n)^{2k}\binom{2n+1}{1}$

$$(2)\qquad \begin{aligned} &s_n(k) - \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}(2n)^2}s_n(k-1) + \cdots \\ &+ (-1)^{k-1}\frac{\binom{2n+1}{2k-1}}{\binom{2n+1}{1}(2n)^{2k-2}}s_n(1) + (-1)^k\frac{\binom{2n+1}{2k+1}k}{\binom{2n+1}{1}(2n)^{2k}} = 0. \end{aligned}$$

Next we take $k=1, 2, 3, \cdots$, successively in (2). As

$$\lim_{n\to\infty}\frac{\binom{2n+1}{2r+1}}{\binom{2n+1}{1}(2n)^{2r}} = \frac{1}{(2r+1)!}$$

for $r=1, 2, \cdots, k$, we see that $\lim_{n\to\infty} s_n(k)$ (exists) $=d_k$ (say), where $d_k (k=1, 2, 3, \cdots)$ is given recursively by

$$(3)\qquad d_k - \frac{d_{k-1}}{3!} + \cdots + (-1)^{k-1}\frac{d_1}{(2k-1)!} = (-1)^{k-1}\frac{k}{(2k+1)!}\cdot$$

A simple inductive argument shows that $|d_k| < 1$, $k=1, 2, \cdots$, so that $\sum_{k=1}^{\infty} d_k x^{2k}$ converges absolutely for $|x| < 1$. Thus using the product theorem for absolutely convergent series we have for $|x| < 1$

$$\begin{aligned} \left\{\sum_{k=1}^{\infty} d_k x^{2k}\right\}\sin x &= \left\{\sum_{k=1}^{\infty} d_k x^{2k}\right\}\left\{\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l+1}}{(2l+1)!}\right\} \\ &= \sum_{m=0}^{\infty}\left\{\frac{d_m}{1!} - \frac{d_{m-1}}{3!} + \cdots + (-1)^{m-1}\frac{d_1}{(2m-1)!}\right\}x^{2m+1} \\ &= \sum_{m=0}^{\infty}(-1)^{m-1}\frac{m}{(2m+1)!}x^{2m+1} \text{ (using (3))} \\ &= \tfrac{1}{2}\{\sin x - x\cos x\} \end{aligned}$$

so that

$$\sum_{k=1}^{\infty} d_k x^{2k} = \frac{1}{2} - \frac{x}{2}\cot x = \frac{1}{2}\sum_{k=1}^{\infty} B_k 2^{2k}\frac{x^{2k}}{2k!}\cdot$$

Equating coefficients we have $d_k=(2^{2k-1}/(2k)!)B_k$, which proves the result.

LEMMA 2.

$$\lim_{n\to\infty}\frac{1}{(2n+1)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)=\frac{1}{\pi^{2k}}\sum_{r=1}^{\infty}\frac{1}{r^{2k}},\qquad k=1,2,\cdots.$$

Proof. The function $\cot^{2k} z$ has a pole of order $2k$ at $z=0$ and is analytic in the annulus $0<|z|<\pi$. By Laurent's theorem there exist complex numbers $a_{-2k}\neq 0, a_{-(2k-1)},\cdots,a_{-1},a_0,a_1,\cdots$ such that

$$\cot^{2k} z=\frac{a_{-2k}}{z^{2k}}+\cdots+\frac{a_{-1}}{z}+a_0+a_1z+\cdots,$$

valid for $0<|z|<\pi$. Clearly $a_{-2k}=\lim_{z\to 0} z^{2k}\cot^{2k} z=(\lim_{z\to 0} z\cot z)^{2k}=1$. Let $a(z)=a_0+a_1z+a_2z^2+\cdots$ so that $a(z)$ is analytic in $|z|<\pi$. Thus in particular $a(z)$ is continuous on the compact set $\{z\mid |z|\leqq\pi/2\}$ and so is bounded there, that is, there is a real number $A(k)\geqq 0$ such that $|a(z)|\leqq A(k)$, for $|z|\leqq\pi/2$. But

$$a(z)=\begin{cases}\cot^{2k} z-\dfrac{a_{-2k}}{z^{2k}}-\cdots-\dfrac{a_{-1}}{z}, & 0<|z|\leqq\pi/2,\\ a_0, \quad z=0,\end{cases}$$

so that

$$\left|\cot^{2k} z-\frac{a_{-2k}}{z^{2k}}-\cdots-\frac{a_{-1}}{z}\right|\leqq A(k),\qquad 0<|z|\leqq\pi/2.$$

Hence there exists a real number $B(k)\geqq 0$ such that

$$\left|\cot^{2k} z-\frac{1}{z^{2k}}\right|\leqq A(k)+\frac{B(k)}{|z|^{2k-1}},\qquad 0<|z|\leqq\pi/2.$$

Taking $z=r\pi/2n+1$ $(r=1,2,\cdots,n)$ we have

$$\left|\cot^{2k}\left(\frac{r\pi}{2n+1}\right)-\frac{(2n+1)^{2k}}{\pi^{2k}r^{2k}}\right|\leqq A+\frac{B(k)(2n+1)^{2k-1}}{\pi^{2k-1}r^{2k-1}},$$

so that

$$\begin{aligned}\left|\frac{1}{2n+1)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)-\frac{1}{\pi^{2k}}\sum_{r=1}^{n}\frac{1}{r^{2k}}\right| &\leqq\sum_{r=1}^{n}\left|\frac{1}{(2n+1)^{2k}}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)\right.\\ \left.-\frac{1}{\pi^{2k}r^{2k}}\right| &\leqq\sum_{r=1}^{n}\left\{\frac{A(k)}{(2n+1)^{2k}}+\frac{B(k)}{\pi^{2k-1}(2n+1)r^{2k-1}}\right\}\\ &\leqq\frac{A(k)n}{(2n+1)^{2k}}+\frac{B(k)}{\pi}\frac{(1+\log n)}{(2n+1)},\end{aligned}$$

as

$$\sum_{r=1}^{n}\frac{1}{r^{2k-1}}\leqq\sum_{r=1}^{n}\frac{1}{r}<1+\int_1^n\frac{dt}{t}=1+\log n.$$

Hence as

$$\lim_{n\to\infty}\left\{\frac{A(k)n}{(2n+1)^{2k}}+\frac{B(k)}{\pi}\,\frac{(1+\log n)}{2n+1}\right\}=0$$

we have

$$\lim_{n\to\infty}\frac{1}{(2n+1)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)=\frac{1}{\pi^{2k}}\sum_{r=1}^{\infty}\frac{1}{r^{2k}}\,.$$

THEOREM. $\sum_{n=1}^{\infty} 1/n^{2k}=(2^{2k-1}\pi^{2k}B_k)/(2k)!$, $k=1, 2, 3, \cdots$.

Proof. This follows immediately from Lemmas 1 and 2 as $\lim_{n\to\infty}(2n/2n+1)^{2k}=1$, for fixed k.

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PRODUCTS OF TRIANGULAR MATRICES

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The following theorem was given in [1] (with an interesting application), and given another proof in [2]. In the summer of 1971 some members of the above-named program gave the proof shown here, and the extension given below.

THEOREM. *Let $S_1, S_2, \cdots, S_n$ be $n\times n$ upper triangular matrices over a ring R such that the (i, i) entry of S_i is 0, then $S_1S_2\cdots S_n=0$.*

For $P=(x_1, x_2, \cdots, x_n)\in R^n$ we have $PS_1=(0, y_2, y_3, \cdots, y_n)$, $PS_1S_2=(0, 0, z_3, z_4, \cdots, z_n), \cdots, PM=0$ where $M=S_1S_2\cdots S_n$. Thus M is the zero map. That $M=0$ follows from taking P to be successively $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots), \cdots$. In case R has no identity we may adjoin an identity, obtaining a ring R' and apply the above argument to R' instead of R.

EXTENSION. The full force of the hypotheses was not used. For example, it is enough to assume about S_1 that its first column is zero, about S_2 that its first two columns are zero after the first term, and so on.

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