AN APPLICATION OF THE OSCILLATION OF A FUNCTION AT A POINT

TOM C. VENABLE, JR., Indiana State University

The purpose of this note is to demonstrate an application of the oscillation of a function f at a point z_0.

DEFINITION. *$w(f, z_0)$ will denote the oscillation of f at the point z_0, i.e., $w(f, z_0) = \inf \sup |f(x)-f(y)|$ where the* sup *is taken over all points x and y in an open neighborhood containing z_0 and the* inf *is taken over all such open neighborhoods.*

It is well known that f is continuous at z_0 if and only if $w(f, z_0)=0$. This concept will be used to obtain information about the set of continuous functions on an interval $[a, b]$; this set will be denoted $C[a, b]$. Let $B[a, b]$ be the set of all bounded functions over $[a, b]$. Using the oscillation concept, it will be shown that $C[a, b]$ is a closed subset of $B[a, b]$ in the uniform topology. A more elementary proof uses the triangle inequality and the fact that a function which is continuous on $[a, b]$ is uniformly continuous on $[a, b]$. [1].

We will show that $B[a, b]-C[a, b]$ is open. Note that this proof uses the direct definition of an open set in a metric space, i.e., a set is open if and only if it is a neighborhood of each of its points. Let $f \in B[a, b]-C[a, b]$. One must show that there exists some sphere about f completely contained in $B[a, b]-C[a, b]$. Since f is not continuous over $[a, b]$, there exists some point $z_0 \in [a, b]$ such that $w(f, z_0)=\delta>0$. Consider the open sphere about f of radius $\delta/4$, i.e., $\{g \mid \sup|f(x)-g(x)|<\delta/4\}$ where x ranges over $[a, b]$. Then $w(g, z_0) \geqq \delta/2$ and g fails to be continuous at z_0 so g is in $B[a, b]-C[a, b]$. Hence, $B[a, b]-C[a, b]$ is open and $C[a, b]$ is closed in $B[a, b]$.

Reference

1. A. E. Taylor, Advanced Calculus, Blaisdell, Waltham, 1955.

NOTE ON $\int_0^\infty (\sin x/x)dx$

KENNETH S. WILLIAMS, Carleton University, Ottawa

In this note we give a simple proof of the well known result $\int_0^{+\infty} (\sin x/x)dx = \pi/2$. (All integrals used are either proper or improper Riemann integrals.) We do this by showing that it is a limiting form of a special case of the following result:

THEOREM 1. *If $f(x)$ is a real-valued function which is continuous on $[a, b]$, where $0 \leqq a < b$, then*

$$\int_0^{+\infty} \left\{ \int_a^b e^{-xy} x f(x)dx \right\} dy \text{ (exists) } = \int_a^b f(x)dx.$$

Proof. Since $f(x)$ is continuous on $[a, b]$, $f(x)$ is bounded on $[a, b]$, say $|f(x)| \leqq M$, for $x \in [a, b]$. Then for $y > 0$ we have

$$\begin{aligned} \left| \int_a^b (1 - e^{-xy}) f(x)dx - \int_a^b f(x)dx \right| &\leqq \int_a^b e^{-xy} \, | f(x) | \, dx \\ &\leqq M \frac{(e^{-ay} - e^{-by})}{y} \\ &\leqq \frac{M}{y}, \end{aligned}$$

so that

$$\lim_{y \to +\infty} \int_a^b (1 - e^{-xy}) f(x)dx = \int_a^b f(x)dx.$$

Now

$$1 - e^{-xy} = \int_0^y x e^{-ux} du,$$

for all x and y so that

$$\begin{aligned} \int_a^b (1 - e^{-xy}) f(x)dx &= \int_a^b \left\{ \int_0^y x e^{-ux} du \right\} f(x)dx \\ &= \int_0^y \left\{ \int_a^b e^{-ux} x f(x)dx \right\} du, \end{aligned}$$

since $e^{-ux} x f(x)$ is continuous on $[a, b] \times [0, y]$. Hence

$$\lim_{y \to +\infty} \int_0^y \left\{ \int_a^b e^{-ux} x f(x)dx \right\} du = \int_a^b f(x)dx,$$

which proves the result.

THEOREM 2. $\int_0^{+\infty} (\sin x/x)dx = \pi/2$.

Proof. In Theorem 1 we choose $a = 0$,

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

so that $f(x)$ is continuous on $[0, b]$, for any $b > 0$. We obtain

$$\int_0^b \frac{\sin x}{x} dx = \int_0^\infty \left\{ \int_0^b e^{-ux} \sin x dx \right\} du$$

$$= \int_0^\infty \frac{1 - e^{-bu}(u \sin b + \cos b)}{1 + u^2} du$$

$$= \frac{\pi}{2} - \int_0^\infty \frac{e^{-bu}(u \sin b + \cos b)}{1 + u^2} du.$$

Now

$$\left| \int_0^\infty \frac{e^{-bu}(u \sin b + \cos b)}{1 + u^2} du \right| \leqq \int_0^\infty \frac{e^{-bu}}{\sqrt{1 + u^2}} du \leqq \int_0^\infty e^{-bu} du = \frac{1}{b},$$

so that letting $b \to +\infty$ we obtain the result.

A DEMOCRATIC PROOF OF A COMBINATORIAL IDENTITY

HARLEY FLANDERS, Tel Aviv University

In a note titled *A new proof of a combinatorial identity*, David C. Shipman gives a matrix theory proof of

$$\sum_{k=j}^{i} \binom{k}{j} \binom{i}{k} (-1)^{j+k} = 0 \quad \text{for } i > j.$$

See this MAGAZINE, 43 (1970) 162–163.

Suppose from an assembly of i individuals one appoints a committee of k and a subcommittee of j. This can be done in

$$\binom{i}{k} \binom{k}{j}$$

ways obviously. However, one could slyly appoint the subcommittee first and then select the other $k-j$ committeemen from the remaining $i-j$ assemblymen. Then the count is

$$\binom{i}{j} \binom{i-j}{k-j}.$$

Thus

$$\sum_{k=j}^{i} \binom{k}{j} \binom{i}{k} x^k = \binom{i}{j} \sum_{k=j}^{i} \binom{i-j}{k-j} x^k$$

$$= \binom{i}{j} x^j \sum_{r=0}^{i-j} \binom{i-j}{r} x^r = \binom{i}{j} x^j (1 + x)^{i-j}.$$

Set $x = -1$.