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A NOTE ON THE QUADRATIC FORM $ax^2 + 2bxy + cy^2$

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The object of this note is to give a simple geometrical proof of the following well-known

Theorem. *Let Λ be a 2-dimensional lattice of determinant $d(\Lambda) \neq 0$. Suppose $a > 0$ and $ac > b^2$. Then there is a point $(u, v) \neq (0, 0)$ of the lattice Λ such that*

$$au^2 + 2buv + cv^2 \leq \frac{2}{\sqrt{3}} \sqrt{(ac - b^2)}\, d(\Lambda)$$

Proof. Suppose all points $(x, y) \neq (0, 0)$ of Λ satisfy

$$ax^2 + 2bxy + cy^2 > \frac{2}{\sqrt{3}} \sqrt{(ac - b^2)}\, d(\Lambda)$$

Let $P = (u, v)$ be one of the lattice points nearest to the origin. Let α be the ellipse $ax^2 + 2bxy + cy^2 = k$, $k > 0$, centre $(0, 0)$, passing through P, so

$$* \qquad au^2 + 2buv + cv^2 = k$$

Let $h = \dfrac{d(\Lambda)}{OP}$ and let l be a line parallel to, and at a distance h from, OP. Suppose l meets α in the 2 points $A = (x_1, y_1)$ and $B = (x_2, y_2)$. The equation of l is

$$vx - uy = \pm h\sqrt{(u^2 + v^2)}$$

Eliminating x between α and l and using * we have that y_1 and y_2 are the roots of

$$ky^2 \pm 2h\sqrt{(u^2 + v^2)}(au + bv)y + (ah^2(u^2 + v^2) - kv^2) = 0$$

$$\therefore y_1 + y_2 = \mp \frac{2h}{k} \sqrt{u^2 + v^2}(au + bv);\ y_1y_2 = \{ah^2(u^2 + v^2) - kv^2\}/k$$

Thus $(y_1 - y_2)^2 = (y_1 + y_2)^2 - 4y_1y_2$

$$= \frac{4v^2}{k^2} [k^2 - (ac - b^2)h^2(u^2 + v^2)] \qquad \text{(using *)}$$

Similarly

$$(x_1 - x_2)^2 = \frac{4u^2}{k^2} [k^2 - (ac - b^2)h^2(u^2 + v^2)]$$

Hence $AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$= \frac{4}{k^2} (u^2 + v^2)[k^2 - (ac - b^2)h^2(u^2 + v^2)]$$

But as α is devoid of lattice points except (0, 0)

$$AB \leq OP$$

and so $k \leq \frac{2}{\sqrt{3}} \sqrt{(ac - b^2)}\, d(\Lambda)$

This contradicts the assumption and the theorem follows.

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