EXERCISES 9, QUESTION 1

1. Let p be a prime such that $p \equiv 3$ or $5 \pmod 8$. Prove that there does not exist an element $\alpha \in O_{\mathbb{Q}(\sqrt{p})}$ such that $N(\alpha) = 2$.

Solution. Suppose there exists $\alpha \in O_{\mathbb{Q}(\sqrt{p})}$, where p is a prime $\equiv 3$ or 5 (mod 8), such that $N(\alpha) = 2$. As $\alpha \in O_{\mathbb{Q}(\sqrt{p})}$, there are integers a and b such that

$$\alpha = \frac{a + b\sqrt{p}}{2}, \; a \equiv b \pmod 2.$$

Now

$$N(\alpha) = N\left(\frac{a+b\sqrt{p}}{2}\right) = \left(\frac{a+b\sqrt{p}}{2}\right)\left(\frac{a-b\sqrt{p}}{2}\right) = \frac{a^2 - pb^2}{4}$$

so

$$a^2 - pb^2 = 8.$$

Hence

$$\left(\frac{2}{p}\right) = \left(\frac{8}{p}\right) = \left(\frac{a^2 - pb^2}{p}\right) = \left(\frac{a^2}{p}\right) = 1,$$

contradicting $p \equiv 3$ or $5 \pmod 8$. Thus no such α exists. ∎

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