EXERCISES 7, QUESTION 2

2. Using the method of Example 7.1.6, prove that $\{1, \sqrt[3]{3}, (\sqrt[3]{3})^2\}$ is an integral basis for $\mathbb{Q}(\sqrt[3]{3})$. What is the discriminant of $\mathbb{Q}(\sqrt[3]{3})$?

Solution. Let $\theta = \sqrt[3]{3}$ and set $K = \mathbb{Q}(\theta) = \mathbb{Q}(\sqrt[3]{3})$. Since θ is a root of the irreducible polynomial $x^3 - 3 \in \mathbb{Z}[x]$ (it is 3-Eisenstein), we have $\theta \in O_K$, $\mathrm{irr}_{\mathbb{Q}}\theta = x^3 - 3$ and $[K : \mathbb{Q}] = \deg(\mathrm{irr}_{\mathbb{Q}}\theta) = 3$. Further

$$D(\theta) = -4 \cdot 0^3 - 27(-3)^2 = -243 = -3^5.$$

As $\dfrac{D(\theta)}{d(K)}$ is a perfect square, we have

$$\frac{D(\theta)}{d(K)} = 1^2,\ 3^2 \text{ or } 3^4$$

so that

$$d(K) = -243, -27 \text{ or } -3.$$

Each of these possibilities is $\equiv 1 \pmod 4$ so we cannot immediately exclude any of them.

We show that

$$O_K = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2,\ d(K) = -243.$$

Clearly

$$\mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2 \subseteq O_K$$

as $\theta \in O_K$. We must show that

$$O_K \subseteq \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2.$$

Let $\alpha \in O_K$. Then $\alpha \in K$ so there exist $x_1, x_2, x_3 \in \mathbb{Q}$ such that

$$\alpha = x_1 + x_2\theta + x_3\theta^2.$$

The K-conjugates of α are

$$\begin{aligned}\alpha &= x_1 + x_2\theta + x_3\theta^2,\\ \alpha' &= x_1 + x_2\omega\theta + x_3\omega^2\theta^2,\\ \alpha'' &= x_1 + x_2\omega^2\theta + x_3\omega\theta^2,\end{aligned}$$

where $\omega^3 = 1,\ \omega \neq 1,\ \omega^2 + \omega + 1 = 0$. Hence

$$\left.\begin{aligned}\alpha + \alpha' + \alpha'' &= 3x_1,\\ \theta^2(\alpha + \omega^2\alpha' + \omega\alpha'') &= 9x_2,\\ \theta(\alpha + \omega\alpha' + \omega^2\alpha'') &= 9x_3,\end{aligned}\right\} \tag{1}$$

as $\theta^3 = 3$. As $\alpha \in O_K$ we have $\alpha,\ \alpha',\ \alpha'' \in \Omega$. As $\theta \in O_K$ we have $\theta, \theta^2 \in \Omega$. As ω, ω^2 are roots of $x^2 + x + 1 \in \mathbb{Z}[x],\ \omega, \omega^2 \in \Omega$. Thus the left hand sides of (1) are in Ω. Hence the right hand sides of (1) are in Ω. They are clearly in $\mathbb{Q}$, so they are in $\Omega \cap \mathbb{Q} = \mathbb{Z}$. Hence

$$9x_1,\ 9x_2,\ 9x_3 \in \mathbb{Z}.$$

Set

$$y_i = 9x_i\ (i = 1, 2, 3)$$

so that $y_i \in \mathbb{Z}\ (i = 1, 2, 3)$ and

$$9\alpha = y_1 + y_2\theta + y_3\theta^2.$$

Clearly (as $3 = \theta^3$) $\theta \mid y_1$ so $3 \mid y_1$. Hence $\theta^2 \mid y_2\theta,\ \theta \mid y_2$ so $3 \mid y_2$. Then $\theta^3 \mid y_3\theta^2,\ \theta \mid y_3$ so $3 \mid y_3$. Set

$$y_i = 3z_i\ (i = 1, 2, 3)$$

so

$$z_i \in \mathbb{Z}\ (i = 1, 2, 3).$$

Then

$$3\alpha = z_1 + z_2\theta + z_3\theta^2.$$

Clearly $\theta \mid z_1$ so $3 \mid z_1$. Hence $\theta^2 \mid z_2\theta,\ \theta \mid z_2$ so $3 \mid z_2$. Then $\theta^3 \mid z_3\theta^2,\ \theta \mid z_3$ so $3 \mid z_3$. Set

$$z_i = 3w_i \ (i = 1, 2, 3)$$

so

$$w_i \in \mathbb{Z} \ (i = 1, 2, 3).$$

Then

$$\alpha = w_1 + w_2\theta + w_3\theta^2 \in \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2.$$

Hence

$$O_K \subseteq \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2$$

so we have proved

$$O_K = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2. \tag{2}$$

Finally

$$\begin{aligned}
d(K) &= D(1, \theta, \theta^2) \ \ \text{(by (2))} \\
&= \begin{vmatrix} 1 & \theta & \theta^2 \\ 1 & \omega\theta & \omega^2\theta^2 \\ 1 & \omega^2\theta & \omega\theta^2 \end{vmatrix}^2 \\
&= \begin{vmatrix} 1 & \theta & \theta^2 \\ 0 & (\omega - 1)\theta & (\omega^2 - 1)\theta^2 \\ 0 & (\omega^2 - 1)\theta & (\omega - 1)\theta^2 \end{vmatrix}^2 \\
&= \begin{vmatrix} (\omega - 1)\theta & (\omega^2 - 1)\theta^2 \\ (\omega^2 - 1)\theta & (\omega - 1)\theta^2 \end{vmatrix}^2 \\
&= (\omega - 1)^4\theta^6 \begin{vmatrix} 1 & (\omega + 1) \\ (\omega + 1) & 1 \end{vmatrix}^2 \\
&= (1 - \omega)^6 3^2 = (-3\omega)^3 3^2 = -3^5.
\end{aligned}$$

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