Chapter 6, Question 21

21. Let $K = \mathbb{Q}(\theta)$, where $\theta^4 - 4\theta^2 + 8 = 0$. Find a rational number c such that $K = \mathbb{Q}(\theta + c\theta^3) \neq K$.

Solution. A simple calculation shows that $x^4 - 4x^2 + 8$ is irreducible in $\mathbb{Q}[x]$. Hence $[K : \mathbb{Q}] = 4$. Suppose that $c \in \mathbb{Q}$ is such that

$$\mathbb{Q}(\theta + c\theta^3) \neq K.$$

Thus

$$\mathbb{Q}(\theta + c\theta^3) \subset K.$$

Hence

$$[\mathbb{Q}(\theta + c\theta^3) : \mathbb{Q}] = 1 \text{ or } 2.$$

Thus $\theta + c\theta^3$ is a root of a quadratic polynomial with rational coefficients. Thus there exist $A, B, C \in \mathbb{Q}$ such that

$$A\left(\theta + c\theta^3\right)^2 + B(\theta + c\theta^3) + C = 0.$$

As

$$\theta^4 = -8 + 4\theta^2,\ \theta^6 = -32 + 8\theta^2,$$

we obtain

$$(C - 16Ac - 32Ac^2) + B\theta + (A + 8Ac + 8Ac^2)\theta^2 + Bc\theta^3 = 0.$$

As $\deg_{\mathbb{Q}}\theta = 4$, $1, \theta\theta^2, \theta^3$ are linearly independent over $\mathbb{Q}$, and so

$$\begin{aligned} C - 16Ac - 32Ac^2 &= 0, \\ B &= 0, \\ A(1 + 8c + 8c^2) &= 0, \\ Bc &= 0. \end{aligned}$$

Since the roots of $1 + 8c + 8c^2 = 0$ are irretional, we deduce that $A = 0$. The first equation then gives $C = 0$. Hence $(A, B, C) = (0, 0, 0)$, a contradiction. This proves there is no rational number c such that $\mathbb{Q}(\theta + c\theta^3) \neq K$. ∎

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