CHAPTER 5, QUESTION 6

6. Prove that $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$.

Solution. Let

$$\alpha = \sqrt{2} + i.$$

Then

$$\alpha^3 = 5i - \sqrt{2}.$$

Hence

$$\alpha + \alpha^3 = 6i, \quad 5\alpha - \alpha^3 = 6\sqrt{2},$$

so that

$$i = \frac{1}{6}\alpha + \frac{1}{6}\alpha^3 \in \mathbb{Q}(\alpha),$$
$$\sqrt{2} = \frac{5}{6}\alpha - \frac{1}{6}\alpha^3 \in \mathbb{Q}(\alpha).$$

Thus

$$\mathbb{Q}(i, \sqrt{2}) \subseteq \mathbb{Q}(\alpha). \tag{1}$$

Clearly

$$\alpha \in \mathbb{Q}(i, \sqrt{2})$$

so

$$\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(i, \sqrt{2}). \tag{2}$$

From (1) and (2) we obtain

$$\mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2} + i). \qquad \blacksquare$$

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