5. Prove that $[\mathbb{Q}(\sqrt{3}+\sqrt[3]{2}):\mathbb{Q}] = 6$, see Example 5.6.2.

Solution. We begin by proving the following result.

Theorem. If K and L are subfields of $\mathbb{C}$ with $K \subseteq L$, and $\alpha \in \mathbb{C}$ is algebraic over K, then

$$\mathrm{irr}_L(\alpha) \mid \mathrm{irr}_K(\alpha) \text{ in } L[x].$$

Proof. As $\alpha \in \mathbb{C}$ is algebraic over K, and $K \subseteq L \subseteq \mathbb{C}$, α is algebraic over L. The minimal polynomial of α over K is $k(x) = \mathrm{irr}_K(\alpha) \in K[x]$ and the minimal polynomial of α over L is $l(x) = \mathrm{irr}_L(\alpha) \in L[x]$. As $K \subseteq L$ we have $k(x) \in L[x]$. As L is a field, $L[x]$ is a unique factorization domain. Thus there exist monic irreducible polynomials $k_1(x), \ldots, k_r(x) \in L[x]$ such that

$$k(x) = k_1(x) \cdots k_r(x).$$

As $k(\alpha) = 0$ we have $k_1(\alpha) \cdots k_r(\alpha) = 0$ so that $k_j(\alpha) = 0$ for some $j \in \{1, 2, \ldots, r\}$. Hence

$$k_j(x) \in I_L(\alpha) =< l(x) >$$

so that

$$l(x) \mid k_j(x) \text{ in } L[x].$$

As $l(x)$ and $k_j(x)$ are both monic and irreducible, we have

$$l(x) = k_j(x).$$

Thus

$$l(x) \mid k(x) \text{ in } L[x].$$

Now

$$\mathrm{irr}_{\mathbb{Q}}(\sqrt[3]{2}) = x^3 - 2$$

so that by the theorem

$$\mathrm{irr}_{\mathbb{Q}(\sqrt{3})}(\sqrt[3]{2}) \mid x^3 - 2 \text{ in } \mathbb{Q}(\sqrt{3})[x].$$

But

$$x^3 - 2 = (x - \sqrt[3]{2})(x - \omega\sqrt[3]{2})(x - \omega^2\sqrt[3]{2}),$$

where $\omega = (-1+\sqrt{-3})/2$, and $\sqrt[3]{2},\ \omega\sqrt[3]{2},\ \omega^2\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{3})$, so that

$$\mathrm{irr}_{\mathbb{Q}(\sqrt{3})}(\sqrt[3]{2}) = x^3 - 2.$$

Thus,

$$[\mathbb{Q}(\sqrt{3}, \sqrt[3]{2} : \mathbb{Q}(\sqrt{3})] = \deg\ (x^3 - 2) = 3.$$

Also

$$\mathrm{irr}_{\mathbb{Q}}(\sqrt{3}) = x^2 - 3$$

so that

$$[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = \deg\ (x^2 - 3) = 2.$$

Thus

$$[\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}) : \mathbb{Q}(\sqrt{3})][\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 3 \cdot 2 = 6.$$

By Example 5.6.2

$$\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}) = \mathbb{Q}(\sqrt{3} + \sqrt[3]{2}).$$

Hence

$$[\mathbb{Q}(\sqrt{3} + \sqrt[3]{2}) : \mathbb{Q}] = 6. \qquad \blacksquare$$

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