CHAPTER 5, QUESTION 18

18. Determine the conjugates of

$$12^{1/5} + 54^{1/5} - 144^{1/5} + 648^{1/5}$$

over $\mathbb{Q}$.

Solution. (Sketch only, for full details see Example 3 in Blair K. Spearman and Kenneth S. Williams, Characterization of solvable quintics, American Mathematical Montly 101 (1994), 986–992.)

$$u_1 = 54^{1/5}, \quad u_2 = 12^{1/5}, \quad u_3 = 648^{1/5}, \quad u_4 = -144^{1/5}.$$

Then,

$$\prod_{j=0}^{4} (x - (\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4)) = x^5 + 330x - 4170,$$

where $\omega = e^{2\pi i/5}$. Hence the conjugates of

$$54^{1/5} + 12^{1/5} + 648^{1/5} - 144^{1/5}$$

are

$$\omega^j 54^{1/5} + \omega^{2j} 12^{1/5} + \omega^{3j} 648^{1/5} - \omega^{4j} 144^{1/5}, \quad j = 0, 1, 2, 3, 4. \qquad \blacksquare$$

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