CHAPTER 5, QUESTION 17

17. Prove that

$$\frac{1}{2}\left(i\sqrt{10+2\sqrt{5}}+2i\sqrt{10-2\sqrt{5}}\right)$$

is an algebraic integer in $\mathbb{Q}(e^{2\pi i/5})$.

Solution. Let

$$\alpha = \frac{1}{2}\left(i\sqrt{10+2\sqrt{5}}+2i\sqrt{10-2\sqrt{5}}\right).$$

Then

$$\begin{aligned} 4\alpha^2 &= \left(i\sqrt{10+2\sqrt{5}}+2i\sqrt{10-2\sqrt{5}}\right)^2 \\ &= -\left(\sqrt{10+2\sqrt{5}}+2\sqrt{10-2\sqrt{5}}\right)^2 = -50-10\sqrt{5}, \end{aligned}$$

as

$$\sqrt{10+2\sqrt{5}}\,\sqrt{10-2\sqrt{5}} = \sqrt{100-20} = \sqrt{80} = 4\sqrt{5}.$$

Hence

$$(4\alpha^2+50)^2 = (-10\sqrt{5})^2 = 500$$

so that

$$\alpha^4 + 25\alpha^2 + 125 = 0.$$

Thus α is a root of a monic polynomial with integral coefficients, and so α is an algebraic integer.

By Questions 15 and 16 we have

$$i\sqrt{10+2\sqrt{5}} = 4\omega - \sqrt{5} + 1 \in \mathbb{Q}(\omega).$$

Similarly

$$i\sqrt{10-2\sqrt{5}} = 4\omega^2 + \sqrt{5} + 1 \in \mathbb{Q}(\omega).$$

Thus

$$\alpha = \frac{1}{2} i \sqrt{10 + 2\sqrt{5}} + i \sqrt{10 - 2\sqrt{5}} \in \mathbb{Q}(\omega).$$

Hence α is an integer of $\mathbb{Q}(\omega)$. $\blacksquare$

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