CHAPTER 5, QUESTION 15

15.Let $\omega = e^{2\pi i/5}$. Prove that

$$\omega = \frac{1}{4}\left(\sqrt{5} - 1 + i\sqrt{10 + 2\sqrt{5}}\right).$$

Solution. Let $\omega = e^{2\pi i/5}$ so that $\omega^5 = e^{2\pi i} = 1$. As $\omega \neq 1$ we have

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = \frac{\omega^5 - 1}{\omega - 1} = 0.$$

Hence

$$(\omega^2 + \frac{1}{2}\omega + 1)^2 - \frac{5}{4}\omega^2 = 0,$$

so that

$$\omega^2 + \left(\frac{1 + \varepsilon\sqrt{5}}{2}\right)\omega + 1 = 0, \quad \varepsilon = \pm 1.$$

Solving the quadratic equation, we obtain

$$\omega = \frac{1}{2}\left(-\left(\frac{1 + \varepsilon\sqrt{5}}{2}\right) + \delta\sqrt{\left(\frac{1 + \varepsilon\sqrt{5}}{2}\right)^2 - 4}\right), \quad \delta = \pm 1,$$

that is

$$\omega = \left(\frac{-1 - \varepsilon\sqrt{5}}{4}\right) + i\left(\frac{\delta\sqrt{10 - 2\varepsilon\sqrt{5}}}{4}\right).$$

Hence

$$\frac{-1 - \varepsilon\sqrt{5}}{4} = \cos\frac{2\pi}{5} > 0$$

so that $\varepsilon = -1$. Then

$$\frac{\delta\sqrt{10 + 2\sqrt{5}}}{4} = \sin\frac{2\pi}{5} > 0$$

so that $\delta = 1$. Thus

$$\omega = \left(\frac{\sqrt{5} - 1}{4} \right) + i \left(\frac{\sqrt{10 + 2\sqrt{5}}}{4} \right). \qquad \blacksquare$$

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