CHAPTER 5, QUESTION 11

11. Determine the conjugates of $3^{1/3} - 3^{2/3}$.

Solution. Let

$$\alpha = 3^{1/3} - 3^{2/3}.$$

Then

$$\alpha^3 = 3 - 9 \cdot 3^{1/3} + 9 \cdot 3^{2/3} - 9 = -6 - 9(3^{1/3} - 3^{2/3}) = -6 - 9\alpha.$$

Hence α is a root of $x^3 + 9x + 6 \in \mathbb{Z}[x]$. This cubic polynomial is 3-Einstein so it is irreducible. Thus it is minimal polynomial of α over $\mathbb{Q}$. The other two conjugates α' and α'' of α are the roots of

$$\frac{x^3 + 9x + 6}{x - \alpha} = x^2 + \alpha x + (\alpha^2 + 9),$$

so that

$$\alpha' = \frac{-\alpha + \sqrt{-36 - 3\alpha^2}}{2}, \quad \alpha' = \frac{-\alpha - \sqrt{-36 - 3\alpha^2}}{2}.$$

Now

$$\alpha^2 + 12 = (3^{1/3} - 3^{2/3})^2 + 4 \cdot 3^{1/3} \cdot 3^{2/3} = (3^{1/3} + 3^{2/3})^2$$

so that

$$\sqrt{\alpha^2 + 12} = 3^{1/3} + 3^{2/3}$$

and thus

$$\sqrt{-36 - 3\alpha^2} = (3^{1/3} + 3^{2/3})\sqrt{-3}.$$

Hence

$$\begin{aligned} \alpha' &= \frac{-(3^{1/3} - 3^{2/3}) + (3^{1/3} + 3^{2/3})\sqrt{-3}}{2} \\ &= 3^{1/3}\left(\frac{-1 + \sqrt{-3}}{2}\right) - 3^{2/3}\left(\frac{-1 - \sqrt{-3}}{2}\right) \\ &= \omega\, 3^{1/3} - \omega^2\, 3^{2/3}. \end{aligned}$$

Similarly

$$\alpha'' = \omega^2 \ 3^{1/3} - \omega \ 3^{2/3}.$$

Thus the three conjugates of $3^{1/3} - 3^{2/3}$ are

$$3^{1/3} - 3^{2/3}, \quad \omega \ 3^{1/3} - \omega^2 \ 3^{2/3}, \quad \omega^2 \ 3^{1/3} - \omega \ 3^{2/3}. \qquad \blacksquare$$

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