EXERCISES 4, QUESTION 4

4. Express the algebraic number

$$\left(\frac{1+\sqrt{2}}{9}\right)^{1/3} + \left(\frac{1-\sqrt{2}}{9}\right)^{1/3}$$

as the quotient of an algebraic integer and an ordinary integer.

Solution. Let

$$\alpha = \left(\frac{1+\sqrt{2}}{9}\right)^{1/3} + \left(\frac{1-\sqrt{2}}{9}\right)^{1/3}.$$

Cubing α, we obtain

$$\begin{aligned}
\alpha^3 &= \left(\frac{1+\sqrt{2}}{9}\right) + 3\left(\frac{1+\sqrt{2}}{9}\right)^{2/3}\left(\frac{1-\sqrt{2}}{9}\right)^{1/3} \\
&\quad +3\left(\frac{1+\sqrt{2}}{9}\right)^{1/3}\left(\frac{1-\sqrt{2}}{9}\right)^{2/3} + \left(\frac{1-\sqrt{2}}{9}\right) \\
&= \frac{2}{9} + 3\left(\frac{1+\sqrt{2}}{9}\right)^{1/3}\left(\frac{1-\sqrt{2}}{9}\right)^{1/3}\left(\left(\frac{1+\sqrt{2}}{9}\right)^{1/3} + \left(\frac{1-\sqrt{2}}{9}\right)^{1/3}\right) \\
&= \frac{2}{9} + 3\left(\frac{(1+\sqrt{2})(1-\sqrt{2})}{81}\right)^{1/3}\alpha \\
&= \frac{2}{9} - \frac{\alpha}{3^{1/3}}
\end{aligned}$$

so that

$$\frac{\alpha}{3^{1/3}} = \frac{2}{9} - \alpha^3.$$

Cubing again we obtain

$$\frac{\alpha^3}{3} = \left(\frac{2}{9} - \alpha^3\right)^3 = \frac{8}{729} - \frac{4}{27}\alpha^3 + \frac{2}{3}\alpha^6 - \alpha^9.$$

Hence

$$243\alpha^3 = 8 - 108\alpha^3 + 486\alpha 66 - 729\alpha^9$$

and so

$$3^6\alpha^9 - 2 \cdot 3^5\alpha 66 + 3^3 \cdot 13\alpha^3 - 8 = 0.$$

Let $\beta = 3\alpha$. Then

$$\begin{aligned}\beta^9 - 18\beta^6 + 351\beta^3 - 216 \\ &= 3^9\alpha^9 - 2 \cdot 3^8\alpha^6 + 3^6 \cdot 13\alpha^3 - 8 \cdot 3^3 \\ &= 3^3(3^6\alpha^9 - 2 \cdot 3^5\alpha^6 + 3^3 \cdot 13\alpha^3 - 8) \\ &= 0.\end{aligned}$$

Hence

$$\beta = 3\left(\left(\frac{1+\sqrt{2}}{9}\right)^{1/3} + \left(\frac{1-\sqrt{2}}{9}\right)^{1/3}\right) = (3+3\sqrt{2})^{1/3} + (3-3\sqrt{2})^{1/3}$$

is an algebraic integer. Finally

$$\alpha = \frac{(3+3\sqrt{2})^{1/3} + (3-3\sqrt{2})^{1/3}}{3},$$

where the numerator is an algebraic integer and the denominator an ordinary integer. ∎

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