CHAPTER 4, QUESTION 15

15. Let A and B be integral domains with $A \subseteq B$ and B integral over A. If I is a nonzero ideal of B, prove that $I \cap A$ is a nonzero ideal of A.

Solution. It is easy to check that $I \cap A$ is an ideal of A. We must show that $I \cap A \neq \{0\}$. As $I \neq \{0\}$ there exist $b \in I$ with $b \neq 0$. Since $b \in B$ and B is integral over A, there exist $a_0, a_1, \ldots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0.$$

Choose n to be the least positive integer for which such a relation holds. If $n = 1$ then $b + a_0 = 0$ so $b = -a_0 \in A$. But $b \in I$ so that $b \in I \cap A$. Hence $I \cap A \neq \{0\}$. Thus we may suppose that $n \geq 2$ so that $n - 1 \geq 1$. If $a_0 = 0$ then $b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1 = 0$, contradicting the minimality of n. Hence $a_0 \neq 0$. Then

$$a_0 = b(-b^{n-1} - \cdots - a_1) \in I.$$

Since $a_0 \in A$ we have $a_0 \in I \cap A$. Hence $I \cap A \neq \{0\}$. ∎

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