CHAPTER 3, QUESTION 17

17. Prove that $\mathbb{Z}+\mathbb{Z}\sqrt{15}$ is not a unique factorization domain.

Solution. Let $D=\mathbb{Z}+\mathbb{Z}\sqrt{15}$. We show that $3, 5, \sqrt{15}$ are irreducibles in D.

Suppose that

$$3=(a+b\sqrt{15})(c+d\sqrt{15}), \quad a,b,c,d\in\mathbb{Z}.$$

Then

$$9=(a^2-15b^2)(c^2-15d^2).$$

Hence

$$a^2-15b^2=\pm1,\pm3,\pm9.$$

We show that $a^2-15b^2\neq\pm3$. Suppose $a^2-15b^2=\pm3$. Then $3\mid a^2$ so $3\mid a$, say $a=3k$. Then $3k^2-5b^2=\pm1$. If the plus sign holds, then

$$1=\left(\frac{1}{5}\right)=\left(\frac{3k^2-5b^2}{5}\right)=\left(\frac{3k^2}{5}\right)=\left(\frac{3}{5}\right)=-1,$$

a contradiction. If the minus sign holds, then

$$1=\left(\frac{1}{3}\right)=\left(\frac{5b^2-3k^2}{3}\right)=\left(\frac{5b^2}{3}\right)=\left(\frac{5}{3}\right)=\left(\frac{-1}{3}\right)=-1,$$

a contradiction. If $a^2-15b^2=\pm1$ then $a+b\sqrt{15}\in U(D)$. If $a^2-15b^2=\pm9$ then $c^2-15d^2=\pm1$ and $c+d\sqrt{15}\in U(D)$. Hence 3 is irreducible in D.

Suppose next that

$$5=(a+b\sqrt{15})(c+d\sqrt{15}).$$

Then

$$25=(a^2-15b^2)(c^2-15d^2).$$

Hence

$$a^2-15b^2=\pm1,\pm5,\pm25.$$

We show that $a^2-15b^2\neq\pm5$. Suppose $a^2-15b^2=\pm5$. Then $5\mid a^2$ so $5\mid a$, say $a=5k$. Then $5k^2-3b^2=\pm1$. As above this equation cannot hold. Hence $a^2 - 15b^2 = \pm 1$ or ± 25 so that $a + b\sqrt{15}$ or $c + d\sqrt{15}$ is a unit, proving 5 is irreducible in D.

In exactly the same way we can show that $\sqrt{15}$ is irreducible in D.

Finally the factorizations

$$15 = 3 \cdot 5 = (\sqrt{15})^2$$

show that D is not a unique factorization domain as $3, 5, \sqrt{15}$ are nonassociated irreducibles of D. ■

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