CHAPTER 3, QUESTION 15

15. Prove that $\mathbb{Z}+\mathbb{Z}\sqrt{-6}$ is not a unique factorization domain by exhibiting an element of $\mathbb{Z}+\mathbb{Z}\sqrt{-6}$ which has two different factorizations into irreducibles.

Solution. Let $D=\mathbb{Z}+\mathbb{Z}\sqrt{-6}$ so that $U(D)=\{\pm 1\}$. We show that $\sqrt{-6}$ is an irreducible in D. Suppose

$$\sqrt{-6}=(a+b\sqrt{-6})(c+d\sqrt{-6}), \quad a,b,c,d\in\mathbb{Z}.$$

Then

$$6=(a^2+6b^2)(c^2+6d^2).$$

Hence

$$a^2+6b^2=1,2,3 \text{ or } 6.$$

If $a^2+6b^2=1$ then $a=\pm 1$, $b=0$ so $a+b\sqrt{-6}=\pm 1$ is a unit. If $a^2+6b^2=6$ then $c^2+6d^2=1$ so $c+d\sqrt{-6}$ is a unit. The possibilities $a^2+6b^2=2,3$ cannot occur. Hence $\sqrt{-6}$ is an irreducible.

Similarly we can show that 2 and 3 are irreducible in D.

Clearly $2,3,\sqrt{-6}$ are not associates of one another.

Hence

$$6=-(\sqrt{-6})^2=2\cdot 3$$

gives two different factorizations of 6 into irreducibles. ∎

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