CHAPTER 2, QUESTION 19

19. Prove that if m is a positive integer possessing a prime divisor $q \equiv 3 \pmod 4$ then there are no integers T and U suct that $T^2 - mU^2 = -1$.

Solution. Suppose that there exist integers T and U suct that $T^2 - mU^2 = -1$, where m is a positive integer possessing a prime divisor $q \equiv 3 \pmod 4$. Then

$$T^2 \equiv T^2 - mU^2 \equiv -1 \pmod q$$

so that

$$\left(\frac{-1}{q}\right) = \left(\frac{T^2}{q}\right) = 1,$$

contradicting $q \equiv 3 \pmod 4$. Hence no such integers T and U exist. ∎

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