CHAPTER 2, QUESTION 18

18. Let p be a prime. Use Theorem 2.5.4 and Question 17 to deduce that

$$p = x^2 + xy + 2y^2 \iff p = 7 \text{ or } p \equiv 1,\ 2,\ 4 \pmod 7.$$

then there do not exist integers x and y such that $p = x^2 + xy + 2y^2$.

Solution. Let p be a prime. If $p = 7$ then $p = x^2 + xy + 2y^2$ with $x = -1$ and $y = 2$. If $p \equiv 1,\ 2,\ 4 \pmod 7$ there exist integers x and y such that $p = x^2 + xy + 2y^2$ by Theorem 2.5.4. If $p \equiv 3,\ 5,\ 6 \pmod 7$ there do not exist integers x and y such that $p = x^2 + xy + 2y^2$ by Question 17. Hence if p is a prime

$$p = x^2 + xy + 2y^2 \iff p = 7 \text{ or } p \equiv 1,\ 2,\ 4 \pmod 7. \qquad \blacksquare$$

June 20, 2004