CHAPTER 2, QUESTION 13

13. Prove that if p is a prime with $p \equiv 5,\ 7 \pmod 8$ then there do not exist integers x and y such that $p = x^2 + 2y^2$.

Solution. Let p be a prime with $p \equiv 5$ or $7 \pmod 8$. Suppose that there exist integers x and y such that

$$p = x^2 + 2y^2.$$

Now $x^2 \equiv 0,\ 1$ or $4 \pmod 8$ and $2y^2 \equiv 0$ or $2 \pmod 8$ so that

$$p = x^2 + 2y^2 \equiv 0 + 0,\ 1 + 0,\ 4 + 0,\ 0 + 2,\ 1 + 2 \text{ or } 4 + 2 \pmod 8,$$

that is

$$p \equiv 0,\ 1,\ 2,\ 3,\ 4 \text{ or } 6 \pmod 8,$$

contradicting $p \equiv 5$ or $7 \pmod 8$. Hence if p is a prime with $p \equiv 5$ or $7 \pmod 8$ there are no integers x and y such that $p = x^2 + 2y^2$. ∎

June 20, 2004