CHAPTER 1, QUESTION 34

34. Let P be a prime ideal of an integral domain D. Let $A_1, \dots, A_k$ be ideals of D such that $P \supseteq A_1 \cdots A_k$. Prove that $P \supseteq A_i$ for some $i \in \{1, 2, \dots, k\}$.

Solution. Let P be a prime ideal of an integral domain D. Let $A_1, \dots, A_k$ be ideals of D such that $P \supseteq A_1 \cdots A_k$. We wish to prove that $P \supseteq A_i$ for some $i \in \{1, 2, \dots, k\}$. If $k = 1$ this is trivially true so we may suppose without loss of generality that $k \geq 2$. We assume that $P \not\supseteq A_i$ for all $i \in \{1, 2, \dots, k\}$ and obtain a contradiction. As

$$P \supseteq (A_1 \cdots A_{k-1})A_k, \ P \not\supseteq A_k,$$

since P is a prime ideal we must have

$$P \supseteq A_1 \cdots A_{k-1}.$$

Then, as

$$P \supseteq (A_1 \cdots A_{k-2})A_{k-1}, \ P \not\supseteq A_{k-1},$$

since P is a prime ideal we must have

$$P \supseteq A_1 \cdots A_{k-2}.$$

Continuing in this way, after a finite number of steps, we obtain

$$P \supseteq A_1,$$

the required contradiction. Hence $P \supseteq A_i$ for some $i \in \{1, 2, \dots, k\}$. ∎

June 19, 2004