16. Prove that $< x >$ is a prime ideal of $\mathbb{Z}[x]$.

Solution. Let $f(x) = f_0 + f_1x + \cdots + f_lx^l \in \mathbb{Z}[x]$ and $g(x) = g_0 + g_1x + \cdots + g_mx^m \in \mathbb{Z}[x]$ be such that

$$f(x)g(x) \in < x > .$$

Then

$$f(x)g(x) = xh(x)$$

for some $h(x) = h_0 + h_1x + \cdots + h_nx^n \in \mathbb{Z}[x]$. Thus

$$(f_0 + f_1x + \cdots + f_lx^l)(g_0 + g_1x + \cdots + g_mx^m) = x(h_0 + h_1x + \cdots + h_nx^n). \quad (1)$$

Equating the constant terms on both sides of (1),we obtain

$$f_0g_0 = 0.$$

As $f_0, g_0 \in \mathbb{Z}$ either $f_0 = 0$ or $g_0 = 0$. If $f_0 = 0$ then $f(x) = f_1x + \cdots + f_lx^l = x(f_1 + \cdots + f_lx^{l-1}) \in < x >$. If $g_0 = 0$ then $g(x) = g_1x + \cdots + g_mx^m = x(g_1 + \cdots + g_mx^{m-1}) \in < x >$. Hence $< x >$ is a prime ideal. ■

June 19, 2004