Balanced Incomplete Block Designs:

Incomplete block designs arise when the number of plots in a block (k) is smaller than the number of treatments (t) so that an RCB design cannot be used.

e.g. Suppose $t = 7$ gasolines are the treatments, cars are blocks and due to lack of time we cannot test each gasoline in each car. In fact, suppose only 3 gasolines can be tested in a car. There are ${}^{7}C_3 = 35$ such possible 3-gasoline combinations and assign the 35 combinations to cars (each car is used for 3 gasolines). Let # of blocks=${}^{7}C_3$.(i.e. thus there are 35 cars needed). Each gasoline appears in ${}^{1}C_1^{7-1}C_{3-1} ={}^{6}\ C_2 = 15$ cars (= r=# of reps per gasoline = # of times a gasoline (i.e. treatment) occurs) and each pair of gasolines occurs in ${}^{2}C_2^{7-2}C_{3-2} ={}^{5}\ C_1 = 5$ cars ($\lambda = \#$ of times a pair of treatments occurs in same block). So we have $b ={}^{t}\ C_k, r ={}^{t-1}\ C_{k-1}, \lambda ={}^{t-2}\ C_{k-2}, t > 2$ (but r will be very large). This gets too large (i.e. too many blocks or cars) if there are a large number of treatments.

An efficient way (i.e. a smaller design) is

A	B	E
C	D	E
A	C	F
B	D	G
A	D	G
B	C	G
E	F	G

where $\lambda = 1, r = 3$, letters denote gasolines and rows denote cars in the above. Here we need 7 (i.e. blocks), each using 3 gasolines. Note: The letters are randomly assigned to the 7 gasolines in the study. This is an example of a balanced incomplete block design.

Definition: B.I.B.D.....t treatments in b blocks.
Each block has $k(< t)$ plots (i.e. pieces of material) and each plot receives a single treatment. Each treatment is replicated r times in the experiment (i.e. appears in r plots). No treatment appears more than once in one block. Each pair of treatments appears in the same block λ times. The five (5) parameters b, t, r, k, λ are not all independent; they are subject to the following relations:
b, t, r, k and λ are integers such that...

"*Necessary conditions*":

1) $N = rt = bk$ (total number of plots)

2) $\lambda(t - 1) = r(k = 1)$ **

3)$b \geq t$ (Fisher's inequality: 1940) (If $b = t$ & $r = k$, we have a symmetric BIBD)

4) $b \geq t + (r - k)$ (sharper than Fisher's)

From ** above: Consider a treatment A which appears in r blocks, each block containing $k - 1$ other treatments. Therefore there are $r(k - 1)$ plots which are in the same blocks as a plot containing A. These plots have to contain the remaining $(t - 1)$ treatments exactly λ times each, since pairs of treatments occur together in the same block λ times..

Proof of (3):

Let $N_{txb} = ((n_{ij}))$ be the incidence matrix, where n_{ij} = # of times treatment i occurs in block j.

For BIBD, $n_{ij} = \begin{cases} 1, \text{ if trt. } i \text{ is in block } j \\ 0, \text{ otherwise} \end{cases}$ $\quad (\therefore\ n_{ij}^2 = n_{ij})$

Let

$$NN^T_{txt} = Q = ((q_{ih}))$$

where

$$\begin{aligned} q_{ii} &= \sum_j n_{ij}^2 \\ &= \sum_j n_{ij} \\ &= r \end{aligned}$$

and

$$\begin{aligned} q_{ih} &= \sum_{j, i \neq h} n_{ij} n_{hj} \\ &= \lambda \end{aligned}$$

Therefore

$$\begin{aligned} NN^T &= Q \\ &= \begin{bmatrix} r & & \lambda \\ & \ddots & \\ \lambda & & r \end{bmatrix} \\ &= (r - \lambda)I + \lambda J \end{aligned}$$

where I is an identity matrix of order t and J is a square matrix of order t with every element 1.
Therefore

$$\begin{aligned} |Q| &= |NN^T| \\ &= (r-\lambda)^{t-1}[r + (t-1)\lambda] \\ &= (r-\lambda)^{t-1}[r + r(k-1)] \\ &= (r-\lambda)^{t-1} rk \\ &> 0 \end{aligned}$$

since $r > \lambda$.
$\therefore\ Q = NN^T$ is non-singular;

$$rank NN^T = rank Q = t$$

$$rank(N) = rank(NN^T) = t = min(b, t)$$

$\Rightarrow b \geq t$ or $r \geq k$.

Proof of (4):

$$(\frac{t}{k} - 1)(r - k) \geq 0$$

using Fisher's inequality since $k < t$,
i.e.

$$\begin{aligned} (\frac{b}{r} - 1)(r - k) &= b - \frac{b}{r}k - (r - k) \\ &= b - t - (r - k) \geq 0 \end{aligned}$$

or

$$b \geq t + (r - k)$$

Proof of (1):
Let E_{1t} =row vector of 1's
Let E_{b1} =column vector of 1's

$$E_{1t}NE_{b1} = bk$$

and

$$E_{1t}N^TE_{b1} = tr$$

Take transpose

$$\begin{aligned} bk &= E_{1t}^T N^T E_{b1}^T \\ &= E_{1b}N^TE_{t1} \end{aligned}$$

Also

$$\begin{aligned} N^TNE_{t1} &= [r + \lambda(t-1)]E_{t1} \\ &= N(N^TE_{t1}) \\ &= rkE_{t1} \end{aligned}$$

Definition: If $b = t$ and $r = k$ ($\Rightarrow$ N square matrix) we have a symmetric *BIBD*.

$$\begin{aligned} |Q| &= |NN'| \\ &= |N|^2 \\ &= (r-\lambda)^{t-1}r^2 \quad \text{is a perfect square} \end{aligned}$$

and $|N| = \pm r(r-\lambda)^{\frac{(t-1)}{2}}$. Since $|N|$ is an integer, $r - \lambda$ is a square when t is even (this is used to show non-existence of a BIBD).

We can use these results to determine if we can construct a BIBD with certain restrictions on the number of blocks, treatments, replication and pairwise replication.
e.g. Is there a BIB with $b = t = 22, r = k = 7, \lambda = 2$?
When $b = t$ and t is even, $r - \lambda$ here is $7 - 2 = 5$ which is not a square, so such a BIBD does not exist.

Construction of BIBD:

1) For any pair $t, k(t > k)$, the simplest thing to do is to take all combinations of k out of t treatments such that $b = {}^tC_k,\ r = {}^{t-1}C_{k-1},\ \lambda = {}^{t-2}C_{k-2}$ but r is very large here.

2) Another is from orthogonal LS where s is prime or power of prime.Then use $b = s(s+1), r = (s+1), k = s, \lambda = 1, t = s^2$.
The $b = s^2 + s$ blocks fall into $s + 1$ sets of s blocks each and each set is a complete replicate (i.e. *resolvable*); the first 2 sets are obtained by listing treatments as $1, ..., s^2$ and first row of the first set is $1, ..., s$; second row is $s+1, ..., 2s$; etc.. For second set, first row is $1, s+1, 2s+1, ...$; second row is $2, s+2, 2s+2, ...$,etc. For third row and so on, continue. Assign treatments coinciding with the letters of LSD.

Analysis of BIBD:

Part 1: Intrablock Analysis (block differences eliminated)

NOTE: ***Estimates of all contrasts in treatment effects can be expressed in terms of comparisons between plots in the same block***.

MODEL: We can write the observation on the i^{th} treatment in the j^{th} block (if there is one) as

$y_{ij} = \mu + \tau_{(i)} + \beta_j + \varepsilon_{ij}$ (i.e. no block-treatment. interaction is assumed)

where μ, τ_i are unknown constants and

$$E(\varepsilon_{ij}) = 0$$
$$V(\varepsilon_{ij}) = \sigma^2$$
$$N = bk = rt$$

Block effects can be fixed or random but fixed block effects will be assumed here.

Minimize the sum of squares of error

$$\phi = \sum_i \sum_j (y_{ij} - [\mu + \beta_j + \tau_{(i)}])^2$$

or consider

$$y_{ij} = \mu + \beta_j + \sum_{s=1}^{t} \delta_{ij}^s \tau_s + e_{ij}, \qquad \text{(where } i \text{ is the plot now)}$$

where $\sum_i \sum_j \delta_{ij}^s = r,$

and $\sum_i \delta_{ij}^s = n_{sj};$

$$\delta_{ij}^s \delta_{ij}^r = 0$$

Normal equations are:

1) $\frac{\partial \phi}{\partial \mu} = 0$:

$$G = \sum_i \sum_j y_{ij} = N\hat{\mu} + k\sum_{j=1}^{b} \hat{\beta}_j + r\sum_{i=1}^{t} \hat{\tau}_i$$

2) $\frac{\partial \phi}{\partial \beta_j} = 0$:

$$B_j = \sum_i y_{ij} = k\hat{\mu} + k\hat{\beta}_j + \sum_{s=1}^{t} n_{sj}\hat{\tau}_s \text{ for each } j$$

3) $\frac{\partial \phi}{\partial \tau_i} = 0$:

$$T_i = r\hat{\mu} + r\hat{\tau}_i + \sum_{j=1}^{b} n_{ij}\hat{\beta}_j \text{ for each } i$$

Additional restrictions: $\sum_j \hat{\beta}_j = \sum_i \hat{\tau}_i = 0$

1)⇒

$$\hat{\mu} = \bar{y}..$$

2)⇒

$$\begin{aligned}\hat{\beta}_j &= \frac{B_j}{k} - \hat{\mu} - \frac{\sum_{s=1}^{t} n_{sj}\hat{\tau}_s}{k} \\ &= \left(\frac{B_j}{k} - \bar{y}..\right) - \frac{\sum_{s=1}^{t} n_{sj}\hat{\tau}_s}{k}\end{aligned}$$

Pay particular attention to how we estimate block effects here.

Substituting this into (3):

$$\begin{aligned}\hat{\tau}_i &= \frac{T_i}{r} - \bar{y}.. - \frac{1}{r}\sum_j n_{ij}\left(\frac{B_j}{k} - \bar{y}.. - \frac{1}{k}\sum_{s=1}^{t} n_{sj}\hat{\tau}_s\right) \\ &= \frac{T_i}{r} - \frac{1}{rk}\sum_{j=1}^{b} n_{ij}B_j + \frac{1}{rk}\sum_{j=1}^{b} n_{ij}\left(\sum_{s=1}^{t} n_{sj}\hat{\tau}_s\right)\end{aligned}$$

Thus,

$$\begin{aligned}rk\hat{\tau}_i &= kT_i - \sum_{j=1}^{b} n_{ij}B_j + \sum_{j=1}^{b} n_{ij}\left(\sum_{s=1}^{t} n_{sj}\hat{\tau}_s\right) \\ &= kT_i - \sum_{j=1}^{b} n_{ij}B_j + \sum_j n_{ij}^2\hat{\tau}_i + \sum_{j=1}^{b} n_{ij}\sum_{s\neq i}^{t} n_{sj}\hat{\tau}_s,\end{aligned}$$

where

$$\sum_j n_{ij}^2\hat{\tau}_i = r\hat{\tau}_i$$

and

$$\sum_{j=1}^{b} n_{ij}\sum_{s\neq i}^{t} n_{sj}\hat{\tau}_s = \lambda\sum_{s\neq i}\hat{\tau}_s$$

i.e.

$$\begin{aligned}(rk - r + \lambda)\hat{\tau}_i &= kT_i - \sum_{j=1}^{b} n_{ij}B_j \\ &= k\left(T_i - \frac{1}{k}\sum_j n_{ij}B_j\right)\end{aligned}$$

Note that

$$\begin{aligned}(rk - r + \lambda) &= r(k-1) + \lambda \\ &= \lambda(t-1) + \lambda \\ &= \lambda t,\end{aligned}$$

and so

$$\begin{aligned}\lambda t\hat{\tau}_i &= kT_i - \sum_j n_{ij}B_j \\ &= k\left(T_i - \frac{1}{k}\sum_j n_{ij}B_j\right)\end{aligned}$$

Define adjusted treatment totals

$$Q_i = \left(T_i - \frac{\sum_j n_{ij} B_j}{k} \right)$$

NOTE:

$$\begin{aligned}\sum Q_i &= \sum T_i - \frac{\sum \sum n_{ij} B_j}{k} \\ &= G - \sum_j B_j \\ &= 0\end{aligned}$$

Therefore

$$kQ_i = \lambda t \hat{\tau}_i$$

or

$$\hat{\tau}_i = \frac{k}{\lambda t} Q_i$$

Now consider that we wish to estimate $\psi = \tau_i - \tau_s$.

Now

$$E(y_{ij} - y_{sj}) = \tau_i - \tau_s$$

so, because ψ can be estimated unbiasedly by a linear combination of the observations, ψ is estimable.

Consider $\hat{\psi} = \hat{\tau}_i - \hat{\tau}_s = \frac{k}{\lambda t}(Q_i - Q_s)$. We show that this is unique minimum variance unbiased estimator (UMVUE) for ψ.

First we obtain the variance of this suggested estimator. Now

$$Var(\hat{\psi}) = (\frac{k}{\lambda t})^2[2Var(Q_i) - 2Cov(Q_i, Q_s)]$$

We need to find $Var(Q_i)$:

We have

$$\begin{aligned} kQ_i &= kT_i - \sum_j n_{ij}B_j - T_i + T_i \\ &= (k-1)T_i - (\sum_j n_{ij}B_j - T_i) \end{aligned}$$

Therefore

$$\begin{aligned} k^2Var(Q_i) &= (k-1)^2Var(T_i) + r(k-1)Var(y_{ij}) \\ &= (k-1)^2r\sigma^2 + r(k-1)\sigma^2 \\ &= r(k-1)k\sigma^2 \end{aligned}$$

thus

$$Var(Q_i) = \frac{r(k-1)}{k}\sigma^2$$

(Note : it does not depend upon *i* so it is the same for all treatments)

We also need to find $Cov\,(Q_i, Q_s)$:

[Note $\sum_i Q_i = 0$ (a constant) and for all pairs $i, s, i \neq s$ from symmetry, $Cov(Q_i, Q_s) = Cov(Q_s, Q_i)$ is the same]

$$0 = Var(\sum_i Q_i) =$$

$$\sum_i Var(Q_i) + \sum_i \sum_{\substack{s \\ i \neq s}} Cov(Q_i, Q_s)$$

$$= \sum_i \frac{r(k-1)}{k}\sigma^2 + t(t-1)Cov(Q_i, Q_s)$$

Therefore

$$
\begin{aligned}
Cov(Q_i, Q_s) &= \frac{-rt(k-1)}{kt(t-1)}\sigma^2 \\
&= \frac{-r}{k}\left(\frac{k-1}{t-1}\right)\sigma^2 \\
&= \frac{-\lambda}{k}\sigma^2
\end{aligned}
$$

Note : This is a constant, regardless of i and s

(NOTE: adjusted treatment totals are not independent)

Thus

$$
\begin{aligned}
Var(\hat{\psi}) &= \frac{2k^2}{\lambda^2 t^2}\left[\frac{r(k-1)}{k}\sigma^2 + \frac{\lambda}{k}\sigma^2\right] \\
&= \frac{2k^2}{(\lambda t)^2}\left(\frac{\lambda t}{k}\right)\sigma^2 = \frac{2k}{\lambda t}\sigma^2
\end{aligned}
$$

The the covariance matrix of Q is given by

$$Cov(Q) = [\lambda t I - \lambda J]\frac{\sigma^2}{k}$$

NOTE: For RBD,

$$Var(\hat{\tau}_i - \hat{\tau}_s) = \frac{2\sigma^2_{RB}}{r}$$

for RB with same # of plots (r observations per treatment).
So, when estimating ψ, the ***efficiency of BIBD over RBD*** is

$$\frac{\textbf{\textit{efficiency of BIBD}}}{\textbf{\textit{efficiency of RBD}}} = \frac{\left(\frac{2\sigma^2_{RB}}{r}\right)}{\left(\frac{2k\sigma^2}{\lambda t}\right)} = \left(\frac{\lambda t}{kr}\right)\frac{\sigma^2_{RB}}{\sigma^2}$$

where $\left(\frac{\lambda t}{kr}\right) < 1$ since

$$
\begin{aligned}
\lambda t &= r(k-1) + \lambda \\
&= rk - (r - \lambda) \\
&< rk
\end{aligned}
$$

In general, for ***any contrast***,

$$\hat{\psi} = \sum_i c_i \hat{\tau}_i = \mathbf{c}'\hat{\boldsymbol{\tau}}, \qquad \sum_i c_i = 0 \;\; (i.e. \mathbf{c}'\mathbf{1} = \mathbf{1}'\mathbf{c} = 0)$$

$$= \frac{k}{\lambda t} \sum_i c_i Q_i$$

(RECALL: adjusted treatment totals are not independent)

$$Var(\hat{\psi}) = Var\left(\sum_i c_i \hat{\tau}_i\right) = \left(\sum_i c_i^2\right)\frac{k}{\lambda t}\sigma^2$$

$$Var_{\text{RBD,with r blocks}}(\hat{\psi}) = \left(\sum_i c_i^2\right)\frac{\sigma_{RB}^2}{r}$$

Suppose we test

$$H_0 : \tau_i = \tau \text{ for all } i \ (\Rightarrow \tau_i = 0 \ \because \ \sum \tau_i = 0)$$

Under the null hypothesis, we get the reduced model

$$y_{ij} = \mu + \beta_j + \varepsilon_{ij} \quad \text{(Model 2)}$$

(which is the model for a CRD where ”treatments” correspond to the blocks)

and we obtain

$$\tilde{\mu} = \bar{y}_{..}$$

$$\tilde{\beta}_j = \left(\frac{B_j}{k} - \bar{y}_{..} \right)$$

The sum of squares for error in the full model (i.e. the BIBD model or Model 1) is

$$R_0^2 = \sum \sum y_{ij}^2 - G\hat{\mu} - \sum_j B_j \hat{\beta}_j - \sum T_i \hat{\tau}_i$$

Recall that

$$\hat{\beta}_j = \left(\frac{B_j}{k} - \bar{y}_{..} \right) - \frac{\sum_{s=1}^t n_{sj} \hat{\tau}_s}{k}$$

and the sum of squares for error in the reduced model (i.e. the CRD model) is

$$\begin{aligned} R_1^2 = R_1^2 &= \sum \sum y_{ij}^2 - G\tilde{\mu} - \sum_j B_j \tilde{\beta}_j \\ &= \sum \sum (y_{.j} - \bar{y}_{..})^2 - \sum_j B_j \left(\frac{B_j}{k} - \bar{y}_{..} \right) \\ &= \sum \sum y_{ij}^2 - \frac{\sum B_j^2}{k} \end{aligned}$$

Thus the reduction in the sum of squares for error by using the more complex design is given by

$$\begin{aligned} R_1^2 - R_0^2 &= -\frac{\sum B_j^2}{k} + \frac{\bar{y}_{..}^2}{N} + \sum_j B_j \left(\frac{B_j}{k} - \bar{y}_{..} - \frac{1}{k} \sum_{s=1}^t n_{sj} \hat{\tau}_s \right) + \sum T_i \hat{\tau}_i \\ &= \sum T_i \hat{\tau}_i - \frac{1}{k} \sum_j B_j \sum_{s=1}^t n_{sj} \hat{\tau}_s = \sum T_i \hat{\tau}_i + \sum_j B_j \hat{\beta}_j - \sum_j \frac{B_j^2}{k} \\ &= \sum_s \left(T_s - \frac{1}{k} \sum_j B_j n_{sj} \right) \hat{\tau}_s = \sum_s Q_s \hat{\tau}_s \\ &= \frac{k}{\lambda t} \sum_s Q_s^2 = S_{t(adj)} \end{aligned}$$

This is the sum of squares for treatments *adjusted for blocks*

(NOTE: While the BIBD design is balanced, treatments are NOT orthogonal to blocks in a BIBD design. We see that the presence of treatments in our model changes the estimator of the $\beta_j's$. i.e. the $\widehat{\beta}_j$ from the full model (or model 1) is not the same as $\widetilde{\beta}_j$ from the reduced model (or model 2) The lack of orthogonality of treatments and blocks complicates the ANOVA table; in particular the sums of squares for treatments, blocks and error do not sum to the total sum of squares).

This leads to an analysis of the experiment, using the ”within block” adjustment for treatments. This is leads to the ***intrablock analysis for BIBD***.

Intrablock analysis for BIBD:

Assuming β_j and ε_{ij} are random normal variables, we obtain

Source	**df**	**SS**	**E(SS)**
Blocks (ignoring treatments)	$b-1$	$\sum_j \frac{B_j^2}{k} - \frac{G^2}{N}$	$(b-1)\sigma_e^2 + (r-\lambda)\sum \frac{\tau_i^2}{k} + (N-k)\sigma_b^2$
Trts (adjusted for blocks)	$t-1$	$\sum_i Q_i \hat{\tau}_i = \frac{k}{\lambda t} \sum_s Q_s^2$	$(t-1)\sigma_e^2 + \lambda t \sum \frac{\tau_i^2}{k}$
Error (intrablock)	$N-b-t+1$	R_0^2	$(N-b-t+1)\sigma_e^2$
Total	$N-1$	$\sum\sum y_{ij}^2 - \frac{G^2}{N}$	

From this table, we can see how to construct a test for the significance of treatment effects.