On Mazur's property and property (X)

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Abstract

We characterize completely those von Neumann algebras whose preduals have Mazur's property. We further show that for preduals of von Neumann algebras, Mazur's property is actually equivalent to property (X) which was first studied by Godefroy and Talagrand in [GOD–TAL 81]. Moreover, we introduce and study natural generalizations of the latter properties to the level of arbitrary cardinal numbers κ, as suggested in [GOD 81] for property (X). In particular, using Edgar's partial ordering of Banach spaces [EDG 83], we prove that property (X) of level κ only differs from the original one in case κ is a measurable cardinal number. Several applications of our results to some concrete spaces such as $L_1(\mathcal{G})$ for a locally compact group $\mathcal{G}$ and the space of trace class operators $\mathcal{T}(\mathcal{H})$ on a Hilbert space are also discussed.

1 Introduction

Our principal aim is to characterize precisely those preduals of von Neumann algebras which enjoy Mazur's property. Moreover, we shall show that for preduals of von Neumann algebras, Mazur's property and property (X) as introduced by Godefroy and Talagrand – which is stronger in general – are actually equivalent. Namely, we shall prove that the predual of a κ-decomposable von Neumann algebra has the latter properties if and only if κ is a non-measurable cardinal. Here, we define κ-decomposability as a natural generalization of the well-known concept of countable decomposability of von Neumann algebras.

This clarifies completely the situation of the above properties in the context of von Neumann algebras. To our knowledge, Mazur's property and property (X) have only been established for *separable* preduals of von Neumann algebras so far – where the first is obtained by a standard argument, the second quite easily. Note that around Mazur's property and property (X) there has been made an unusual number of errors in the literature, as we shall briefly indicate later. In particular, we have to stress here that in contrast to what is claimed at different places in the literature, it is *not* true that the predual of any von Neumann algebra has Mazur's property or even property (X). A counterexample has been pointed out by Edgar.

Theorem 1.1. *For an abstract set Γ, the space $\ell_1(\Gamma)$ has Mazur's property if and only if it has property (X), which in turn happens if and only if $|\Gamma|$ is a non-measurable cardinal.*

Proof. For these two results, see [EDG 79], Thm. 5.10 and [EDG 83], Prop. 12. □

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Even if one assumes that measurable cardinals do not exist at all, no proof is known to the author which would establish property (X) or at least Mazur's property for preduals of von Neumann algebras. We should mention nevertheless that it is claimed in [EDG 83], p. 90 that Godefroy would prove the latter assertion in [GOD 81], but this is not clear to us – an unclarity that is also shared by Gilles Godefroy himself, as we know from private communication. We already remark at this point that, by proving property (X) in our general situation, we will also find another quite large class of Banach spaces having property (X) (see Theorem 3.19 below).

We will further introduce and study natural generalizations of Mazur's property and property (X) for an arbitrary cardinal number κ. We remark that it has been proposed in the "Erratum" added to [GOD 81] to investigate such a variant of property (X) – but to our knowledge, nobody has taken this route so far. In doing so, we shall prove the surprising result that this new property only differs from the classical one in case the cardinal number involved is measurable. This will be obtained by using the ordering of Banach spaces introduced by Edgar in [EDG 83].

Our original interest was stemming from the question whether the space $L_1(\mathcal{G})$, where $\mathcal{G}$ denotes an arbitrary locally compact group, has Mazur's property or even property (X). By the above mentioned theorem, this is not true in general. So the question was to find out under which circumstances the answer is positive and how one could change the property in order to obtain a variant which in turn would hold for *any* locally compact group. We were also interested in having some normality criteria at hand also in the case of the non-commutative counterpart of the above, namely the trace class operators $\mathcal{T}(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space (e.g., $\mathcal{H} = L_2(\mathcal{G})$). All these questions will turn out to be special cases of the theorems we shall prove, and the corresponding answers will thus serve as applications of the more general results.

The paper is organized as follows. First we shall provide the background from the general theory of topological groups necessary for the main applications of our results in some concrete situations occurring in abstract harmonic analysis. We briefly note a useful, explicit description of the compact covering number of an arbitrary locally compact group. The third section then establishes the equivalence of Mazur's property and property (X) of the predual $\mathcal{M}_*$ as well as the κ-decomposability for non-measurable κ for an arbitrary von Neumann algebra $\mathcal{M}$. In the last section we discuss the generalizations of both properties to the level of arbitrary cardinal numbers as briefly sketched above.

We would like to stress that the concepts and results contained in this paper have recently led to various important applications in Banach algebra theory and abstract harmonic analysis. – Dales and Lau [DAL–LAU 04] as well as the author [NEU 04e] have used results such as Theorem 3.24 and Theorem 4.4, respectively, in their investigation of the topological centres of the second dual of Beurling algebras. Moreover, the notion of decomposability introduced and studied in section 3 below has been of fundamental importance in the recent article [HU–NEU 04] by Hu and the author, and in Hu's subsequent papers [HU 04] and [HU 04], the latter completing the programme started in [HU 02]. Finally, the author has used Theorem 4.4 in his new approach to strong Arens irregularity of group algebras [NEU 04c], as well as in his recent proof [NEU 04b] of the Hofmeier–Wittstock conjecture [HOF–WIT 97] concerning the automatic boundedness of linear (left) $L_\infty(\mathcal{G})^*$-module maps on $L_\infty(\mathcal{G})$. Also he has used Theorem 3.12 in his recent proof [NEU 04d] of the Ghahramani–Lau conjecture (see [LAU 94] and [GHA–LAU 95]) on the topological centre of the second dual of the measure algebra $M(\mathcal{G})$. Last but not least, Theorem 3.12 has also proved useful in his study [NEU 04a] of amplifications of completely bounded operators on von Neumann algebras.

2 The compact covering number of a locally compact group

For every (non-empty) locally compact Hausdorff space, the minimal cardinality of a compact covering of X is known as the compact covering number of X (cf., e.g., [COM 84], §3, p. 1163). We shall denote this cardinality by $\mathfrak{k}(X)$. Now let $\mathcal{G}$ be a locally compact (Hausdorff) group. We wish to show that in this situation, one has a concrete and explicit description of the cardinality $\mathfrak{k}(\mathcal{G})$.

Theorem 2.1. *Let $\mathcal{G}$ be a locally compact group. For every open σ-compact subgroup $\mathcal{H}$ of $\mathcal{G}$ we have the equality:*

$$\mathfrak{k}(\mathcal{G}) \cdot \aleph_0 = |\mathcal{G}/\mathcal{H}| \cdot \aleph_0.$$

Proof. Cf. [COM 84], §3, Lemma 3.11, p. 1167; there, $\mathcal{H}$ in addition is assumed to be compactly generated. Inspection of the proof shows that this assumption is unnecessary. For the sake of completeness, however, we include a transparent proof of the above more general statement.

"$\leq$": There exist compact subsets $K_n \subseteq \mathcal{G}$ ($n \in \mathbb{N}$) such that $\mathcal{H} = \bigcup_{n\in\mathbb{N}} K_n$. We thus obtain (choosing once arbitrary representatives $x \in \mathcal{G}$ of the respective class $[x] = x\mathcal{H}$):

$$\mathcal{G} = \dot{\bigcup_{[x]\in\mathcal{G}/\mathcal{H}}} x\mathcal{H} = \dot{\bigcup_{[x]\in\mathcal{G}/\mathcal{H}}} \bigcup_{n\in\mathbb{N}} xK_n = \bigcup_{([x],n)\in\mathcal{G}/\mathcal{H}\times\mathbb{N}} xK_n.$$

Hence, $(xK_n)_{([x],n)\in\mathcal{G}/\mathcal{H}\times\mathbb{N}}$ is a compact covering of $\mathcal{G}$. We conclude that

$$\mathfrak{k}(\mathcal{G}) \cdot \aleph_0 \leq |\mathcal{G}/\mathcal{H} \times \mathbb{N}| \cdot \aleph_0 = |\mathcal{G}/\mathcal{H}| \cdot \aleph_0.$$

"$\geq$": Let $(K_\alpha)_{\alpha\in I}$ be a compact covering of $\mathcal{G}$ with $|I| = \mathfrak{k}(\mathcal{G})$. We write:

$$\mathcal{G} = \dot{\bigcup_{[x]\in\mathcal{G}/\mathcal{H}}} x\mathcal{H}.$$

Since every K_α, $\alpha \in I$, is compact and covered by the open sets $x\mathcal{H}$, $x\mathcal{H} = [x] \in \mathcal{G}/\mathcal{H}$, to each $\alpha \in I$ there exist finitely many elements $x_{(\alpha,1)}, \ldots, x_{(\alpha,n_\alpha)} \in \mathcal{G}$ such that $K_\alpha \subseteq \bigcup_{i=1}^{n_\alpha} x_{(\alpha,i)}\mathcal{H}$. For $m > n_\alpha$, put $x_{(\alpha,m)} := x_{(\alpha,1)}$. Hence we see that

$$\mathcal{G} = \bigcup_{\alpha\in I} K_\alpha = \bigcup_{\alpha\in I} \bigcup_{i\in\mathbb{N}} x_{(\alpha,i)}\mathcal{H}.$$

We thus have:

$$\mathcal{G} = \bigcup_{(\alpha,i)\in I\times\mathbb{N}} x_{(\alpha,i)}\mathcal{H} = \dot{\bigcup_{[x]\in\mathcal{G}/\mathcal{H}}} x\mathcal{H}.$$

These are two coverings of $\mathcal{G}$ with the same sets where the latter is disjoint. Hence, to every $x\mathcal{H} = [x] \in \mathcal{G}/\mathcal{H}$ there exists an index $(\alpha, i) \in I \times \mathbb{N}$ such that $x\mathcal{H} = x_{(\alpha,i)}\mathcal{H}$. Therefore, the following map is well-defined:

$$\begin{array}{rcl} \Phi : \mathcal{G}/\mathcal{H} & \longrightarrow & I \times \mathbb{N} \\ x\mathcal{H} & \mapsto & (\alpha, i) \quad \text{where } x\mathcal{H} = x_{(\alpha,i)}\mathcal{H}. \end{array}$$

Since Φ obviously is injective, we have:

$$|\mathcal{G}/\mathcal{H}| \cdot \aleph_0 \leq |I \times \mathbb{N}| \cdot \aleph_0 = \mathfrak{k}(\mathcal{G}) \cdot \aleph_0,$$

which ends the proof. $\square$

Corollary 2.2. *Let $\mathcal{G}$ be a locally compact, non-σ-compact group. Then for every open σ-compact subgroup $\mathcal{H}$ of $\mathcal{G}$ we have the equality $\mathfrak{k}(\mathcal{G}) = |\mathcal{G}/\mathcal{H}|$.*

3 Mazur's property and property (X) for preduals of von Neumann algebras

We shall begin by briefly recalling the definition of Mazur's property as well as of the (stronger) property (X) of a Banach space Y. In both cases, the property enables one to characterize completely the normality of functionals on Y^* only by their *sequential* behaviour – this means that to know whether a functional is normal or not it suffices to "test it against sequences", instead of considering, as usual, nets of arbitrary cardinality.

Definition 3.1. *Let Y be a Banach space. Then Y is said to have Mazur's property (or to satisfy the condition of Mazur) if the following criterion about normality holds: A functional in Y^{**} is normal (i.e., defines an element of Y) if and only if it is w^*-sequentially continuous (on the unit ball of Y^*).*

We remark that spaces with Mazur's property sometimes are also called d-complete (cf. [KAP 86]) or μB spaces (cf. [WIL 81]).

Property (X) has been introduced in [GOD 81], Déf. 5, in connection with the question of unicity of preduals (up to isometry) of dual Banach spaces (cf. also [GOD–TAL 81], Déf. 3; [GOD 89], p. 158, condition $(**)$ in Thm. V.3; [HAR–WER–WER 93], p. 147). To formulate the latter property, we briefly recall the notion of weakly unconditionally Cauchy (wuC) series, also called weakly unconditionally convergent (wuc) series in the literature (cf. [GOD 89], p. 157–158). We follow the terminology of [HAR–WER–WER 93], III.3.3, p. 127.

Definition 3.2. *Let X be a Banach space. A series $\sum f_n$ in X is called weakly unconditionally Cauchy (wuC) if for every functional $\Phi \in X^*$ we have:*

$$\sum_{n=1}^{\infty} |\langle \Phi, f_n \rangle| < \infty.$$

Now, still following [HAR–WER–WER 93], III.5, p. 147, we come to

Definition 3.3. *A Banach space Y is said to have property (X) if the following normality criterion is true: If $f \in Y^{**}$ is a functional such that for every wuC series $\sum y_n$ in Y^*, the equality*

$$\left\langle f, w^* - \sum y_n \right\rangle = \sum \langle f, y_n \rangle$$

holds, then we have $f \in Y$. (Here, the limit $w^* - \sum y_n$ is taken in the $\sigma(Y^*, Y)$-topology.)

For later purposes, we collect some basic properties of the above properties.

Remark 3.4. *As is very easily seen, Mazur's property as well as property* (X) *are stable under isomorphism. They are also hereditary, i.e., they pass to (closed) subspaces. (In both cases, this can be obtained in essentially the same fashion as one proves that reflexivity is hereditary.)*

For these statements (without proof), see [KAP 86], Prop. 2.2 (1) and p. 625 or [LEU 91], p. 51 concerning Mazur's property and Prop. 4 in [GOD–TAL 81] for property (X).

Remark 3.5. *We briefly point out examples of spaces which have Mazur's property but not property* (X). *To this end, we recall that if a Banach space* X *has property* (X), *then* X *is weakly sequentially complete (see [GOD 89], Remark following Thm. V.3, p. 159). Hence, the spaces* c_0 *and* $\mathcal{K}(\mathfrak{H})$, *where* $\mathfrak{H}$ *is an infinite-dimensional separable Hilbert space, do not have property* (X), *but being separable, they have Mazur's property.*

In fact, if a Banach space X *has property* (X), *then* X *is the unique isometric predual of* X^* *(see [GOD 89], Thm. V.3 (i)); clearly Mazur's property does not imply the latter.*

Let us finally mention a property which is intermediate between Mazur's property and property (X) and which has been introduced as "property $(*)$" by Godefroy (cf. [GOD 89], p. 155). A Banach space Y is said to have property $(*)$ if the following holds true: A functional $f \in Y^{**}$ actually belongs to Y if for every weakly Cauchy sequence $(y_n) \subseteq Y^*$, one has $\langle f, w^* - \lim y_n\rangle = \lim\langle f, y_n\rangle$.

Our investigations will start with a

Definition 3.6. *Let* κ *be a cardinal number. A von Neumann algebra* $\mathcal{M}$ *will be called* κ*-decomposable if any family of pairwise orthogonal non-zero projections in* $\mathcal{M}$ *has at most cardinality* κ.

Remark 3.7. *Of course, in case* $\kappa = \aleph_0$, *this is the usual notion of countable decomposability (or* σ*-finiteness).*

First, just as in the classical setting, we will show the connection between decomposability and the existence of separating families for $\mathcal{M}$.

Proposition 3.8. *Let* $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ *be a von Neumann algebra and* $\kappa \geq \aleph_0$ *a cardinal number. Then* $\mathcal{M}$ *is* κ*-decomposable if and only if there exists a family* $(\zeta_i)_{i\in I} \subseteq \mathcal{H}$ *with* $|I| \leq \kappa$ *which is separating for* $\mathcal{M}$ *(i.e., cyclic for* $\mathcal{M}'$, *the commutant of* $\mathcal{M}$*).*

Proof. Suppose first that $\mathcal{M}$ is κ-decomposable, and let $(\zeta_i)_{i\in I}$ be a maximal family of non-zero vectors in $\mathcal{H}$, such that the spaces $\overline{\mathcal{M}'\zeta_i}$ and $\overline{\mathcal{M}'\zeta_j}$ are orthogonal for $i, j \in I$ with $i \neq j$. Since the projections e_i from $\mathcal{H}$ onto $\overline{\mathcal{M}'\zeta_i}$ are elements of $\mathcal{M}$ and are pairwise orthogonal, we deduce from our assumption that $|I| \leq \kappa$. Maximality of the family $(\zeta_i)_{i\in I}$ guarantees that $\bigoplus_{i\in I} \overline{\mathcal{M}'\zeta_i} = \mathcal{H}$, hence, $(\zeta_i)_{i\in I}$ is cyclic for $\mathcal{M}'$.

To prove the second implication, let $(\zeta_i)_{i\in I}$ be a family which is separating for $\mathcal{M}$, where $|I| \leq \kappa$. Let further $(p_l)_{l\in L} \subseteq \mathcal{M}$ be an arbitrary family of orthogonal non-zero projections. Since for all $i \in I$:

$$\sum_{l\in L} \|p_l\zeta_i\|^2 \leq \|\zeta_i\|^2,$$

we clearly have that for each $i \in I$, there is an index set $J_i \subseteq L$ which is at most countable and such that $p_l\zeta_i = 0$ for all $l \notin J_i$. Putting $J := \bigcup_{i\in I} J_i \subseteq L$, we thus get an index set of cardinality $|J| \leq \kappa$ such that $p_l\zeta_i = 0$ for all $i \in I$ and $l \notin J$. Finally, $(\zeta_i)_{i\in I}$ being separating for $\mathcal{M}$, we obtain $L = J$. $\square$

Corollary 3.9. *Let κ be a cardinal number. The von Neumann algebra $\mathcal{B}(\mathcal{H})$ is κ-decomposable if and only if* $\dim(\mathcal{H}) \leq \kappa$, *where* dim *denotes the Hilbert space dimension.*

Proof. The assertion is clear in case κ is finite. Now if $\kappa \geq \aleph_0$, one has just to take into account the fact that $\mathcal{B}(\mathcal{H})' = \mathbb{C}\ \mathbb{1}$ and to apply Proposition 3.8. $\square$

We now recall a

Definition 3.10. *A cardinal number κ is called (real-valued) measurable if for every abstract set Γ of cardinality κ there exists a probability measure on the power set $\mathfrak{P}(\Gamma)$ which is diffused, i.e., vanishes on singletons.*

Obviously, measurability is a property of "large" cardinals. We remark that, if κ_1 and κ_2 are two cardinal numbers with $\kappa_1 \leq \kappa_2$, and if κ_2 is non-measurable, then of course κ_1 also is non-measurable.

Remark 3.11. *We stress the fact that the existence of measurable cardinals cannot be proved in ZFC (= the axioms of Zermelo-Fraenkel and the axiom of choice); for this, cf. [KAN–MAG 78], §1, p. 106 & 108. On the other hand, it is consistent with ZFC to assume that measurable cardinals do not exist ([GAR–PFE 84], §4, Thm. 4.14, p. 972).*

As for the somehow pathological nature of the assumption of the existence of measurable cardinals, we note that (cf. [GAR–PFE 84], §4, Thm. 4.14, p. 972) in ZFC the following statements are equiconsistent (i.e., if one is consistent with ZFC, so is the other one): "there exists a measurable cardinal" and "Lebesgue measure admits a σ-additive extension on the power set $\mathfrak{P}(\mathbb{R})$ (which will not be translation invariant of course)". We finish our brief discussion of measurable cardinals by remarking that, to give a concrete example, there do not exist measurable cardinals in ZFC + "V=L", where the latter denotes Gödel's constructibility axiom ([SCO 61], p. 521). – The reader will find detailed information about measurable cardinals in the articles [FRE 93] and [SOL 71].

Our first main goal is the following

Theorem 3.12. *Let $\mathcal{M}$ be a von Neumann algebra. Then the following are equivalent:*

(i) The predual $\mathcal{M}_*$ of $\mathcal{M}$ has Mazur's property.

(ii) The von Neumann algebra $\mathcal{M}$ is κ-decomposable for some non-measurable cardinal number κ.

We will now present some auxiliary results preparing the proof of Theorem 3.12. We formulate them separately since we shall also use them later on in our study of property (X).

Lemma 3.13. *Let $\mathcal{M}$ be a von Neumann algebra. A linear form on $\mathcal{M}$ is normal if and only if its restriction to every abelian von Neumann subalgebra of $\mathcal{M}$ is normal.*

Proof. See [TAK 58], Cor. 1. $\square$

For the following, we recall that a Banach space is said to be weakly compactly generated (WCG) if it has a weakly compact subset whose linear span is dense. A Banach space X is weakly countably determined (WCD) or a Vašák space if there is a sequence $(K_n)_{n\in\mathbb{N}} \subseteq X^{**}$ of w^*-compact sets such that for $x \in X$ and $u \in X^{**} \setminus X$ there is $n \in \mathbb{N}$ such that $x \in K_n$ and $u \notin K_n$. Finally, a Banach space X is called weakly Lindelöf determined (WLD) if $\mathrm{Ball}(X^*)$ in its w^*-topology is a Corson compact (i.e., homeomorphic to a subset of $\{x \in [-1,1]^\Gamma : |\{\gamma \in \Gamma : x_\gamma \neq 0\}| \leq \aleph_0\}$ for some set Γ). We stress that every WCG Banach space is WCD, and every WCD Banach space in turn is WLD, but none of the converse assertions holds (cf. [Ziz 03], Thm. 3.8).

For a detailed discussion of these classes of Banach spaces, we refer to the excellent survey paper [Ziz 03]. The author is grateful to Václav Zizler for discussions about several properties of WLD spaces and also for providing him with an early copy of [Ziz 03].

Lemma 3.14. *Every WLD Banach space has Mazur's property.*

Proof. If X is a WLD space, by definition, $(\mathrm{Ball}(X^*), w^*)$ is a Corson compact, hence an angelic space (see [Neg 84], Remarks 6.3 (c), p. 1100). Owing to [Kap 86], Prop. 2.3 (cf. also [Edg 79], p. 564, Figure 1), this implies that X has Mazur's property. □

Example 3.15. *Let (Ω, μ) be a σ-finite measure space. Then $L_1(\Omega, \mu)$ is weakly compactly generated and therefore enjoys Mazur's property.*

Proof. Following [Lin 72], Cor. 2 (or [Hab–Háj–Ziz 96], Ch. 10, p. 217, Examples), the space $L_1(\Omega, \mu)$ is weakly compactly generated precisely in case (Ω, μ) is σ-finite. □

Lemma 3.16. *Let I be an abstract set whose cardinality is non-measurable. Let further $(Y_i)_{i\in I}$ be a family of Banach spaces*
(i) with Mazur's property
(ii) with property (X).
Then the space $Y = \bigoplus_{i\in I}^{\ell_1} Y_i$ also has the respective property.

Proof. Assertion (i) is Thm. 3.1 in [Kap 86], assertion (ii) is Prop. 15 in [Edg 83]. – As for (i), we remark that the argument in [Kap 86] contains a little error: the choice made there of elements $y_i \in X_i$ is not possible in general. The proof can of course be repaired; but we prefer to refer to Thm. 4.2 in [Leu 91] where an even more general result is obtained by a completely different method. □

We are now ready for the proof of Theorem 3.12.

Proof. *(i)* ⇒ *(ii)*: Assuming the contrary, let $(P_\alpha)_{\alpha\in I}$ be a family of pairwise orthogonal, non-zero projections in $\mathcal{M}$, where $|I|$ is a measurable cardinal. Now let $\mathcal{R}$ be the von Neumann algebra in $\mathcal{M}$ generated by these projections. Then $\mathcal{R}$ is an abelian atomic von Neumann subalgebra of $\mathcal{M}$. Hence, by [Tom 70], p. 23–24, there exists a normal projection of norm 1 from $\mathcal{M}$ onto $\mathcal{R}$. We thus see that the predual $\mathcal{R}_*$ is isometrically isomorphic to a (closed) subspace of the predual $\mathcal{M}_*$. Since the latter has Mazur's property, which is both hereditary and stable under isomorphism (see Remark 3.4), we deduce that $\mathcal{R}_*$ also has Mazur's property.

Now, by a standard argument (cf. [Kad–Rin 86], p. 669), we see that

$$\mathcal{R} = \left\{ \sum_{\alpha\in I} f_\alpha P_\alpha \mid (f_\alpha) \in \ell_\infty(I) \right\}.$$

(In fact, by minimality of P_α we have for arbitrary $A \in \mathcal{R}$ that $AP_\alpha = P_\alpha A P_\alpha = f_\alpha P_\alpha$ with $f_\alpha \in \mathbb{C}$, $|f_\alpha| \leq \|A\|$; hence, since $\mathbb{1}_{\mathcal{R}} = \sum P_\alpha$, we obtain $A = \sum AP_\alpha = \sum f_\alpha P_\alpha$.) Consider the mapping

$$\begin{array}{rcl} \pi : \ell_\infty(I) & \longrightarrow & \mathcal{R} = \left\{ \sum\limits_{\alpha \in I} f_\alpha P_\alpha \mid (f_\alpha) \in \ell_\infty(I) \right\} \\ f & \mapsto & \sum\limits_{\alpha \in I} f_\alpha P_\alpha . \end{array}$$

Obviously, π is a *-isomorphism from $\ell_\infty(I)$ onto $\mathcal{R}$, hence normal (and isometric). We conclude that $\mathcal{R}_*$ and $\ell_1(I)$ are isomorphic as Banach spaces. Since $\mathcal{R}_*$ has Mazur's property, the same is true for $\ell_1(I)$. But this implies ([EDG 79], Thm. 5.10) that $|I|$ is a non-measurable cardinal – which is the desired contradiction.

(ii) $\Rightarrow$ *(i)*: Let $\mathcal{M}$ be a κ-decomposable von Neumann algebra. Thanks to Lemma 3.13, it is easily seen that we can restrict ourselves to the case where $\mathcal{M}$ is an abelian von Neumann algebra.

We follow the discussion of abelian W^*-algebras presented in [SAK 98], 1.18, in particular Prop. 1.18.1 (with proof), as well as the terminology used there. For a finite measure space (Ω, μ), we denote by $L_\infty(\Omega, \mu)$ the abelian W^*-algebra of essentially bounded μ-measurable functions on Ω. In the proof of [SAK 98], Prop. 1.18.1, it is shown that for an abelian von Neumann algebra, and so for $\mathcal{M}$, we have:

$$\mathcal{M} = \bigoplus_{\alpha \in I}^{\ell_\infty} L_\infty(\Omega_\alpha, \mu_\alpha)$$

as von Neumann algebras, where $(\Omega_\alpha, \mu_\alpha)$ are finite measure spaces. We thus get an isometric isomorphism:

$$\mathcal{M}_* = \bigoplus_{\alpha \in I}^{\ell_1} L_1(\Omega_\alpha, \mu_\alpha).$$

By Example 3.15, the spaces $L_1(\Omega_\alpha, \mu_\alpha)$ have Mazur's property. Since obviously $|I| \leq \kappa$, and κ is non-measurable, we see that $|I|$ also is non-measurable. Hence, noting that Mazur's property is stable under isomorphism (see Remark 3.4), we deduce from Lemma 3.16 (i) that $\mathcal{M}_*$ has Mazur's property, as desired. □

We close the discussion of Mazur's property with pointing out that unfortunately, in connection with the latter – and also with property (X) – various errors have been made in the literature. For instance, some statements in this context from [DIE 76], p. 221 & 223, are not correct in the generality stated there. This concerns the discussion of Mazur's property for L_1-spaces (p. 221 & 223), where one has to take into account Theorem 1.1, and for spaces whose (closed) dual unit ball is w^* sequentially compact (p. 221); for the latter, we refer to Schachermayer's example which is presented, e.g., in [WIL 81], p. 49.

We will now definitively let property (X) enter our discussion. To begin with, we again stress that, as shown by Edgar's example (Thm. 1.1) and also by Theorem 3.12, as soon as one accepts the existence of measurable cardinals, not every predual of a von Neumann algebra has property (X). Whenever the contrary was claimed in the literature, probably the Banach space considered was tacitly assumed to be separable ([GOD–TAL 81], Thm. 6 or [GOD 89], Example (4), p. 161; cf. also the "Erratum" added to [GOD 81]). – We now state one of our main theorems.

Theorem 3.17. *Let $\mathcal{M}$ be a von Neumann algebra. Then the following are equivalent:*

- *(i) The predual $\mathcal{M}_*$ of $\mathcal{M}$ has property (X) of Godefroy-Talagrand.*
- *(ii) The predual $\mathcal{M}_*$ of $\mathcal{M}$ has property $(*)$ from [GOD 89].*
- *(iii) The predual $\mathcal{M}_*$ of $\mathcal{M}$ has Mazur's property.*
- *(iv) The von Neumann algebra $\mathcal{M}$ is κ-decomposable for some non-measurable cardinal number κ.*

The proof will (of course) follow the scheme "$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$". Here, with Theorem 3.12 at hand, we need only to show the implication "$(iv) \Rightarrow (i)$". This in turn will of course yield another approach to the implication "$(ii) \Rightarrow (i)$" of Theorem 3.12. But since the proof of the stronger statement, as one should expect, is more complicated, we found it valuable to present a self-contained proof of Theorem 3.12.

We shall now present the preparation required by the proof of Theorem 3.17. But let us before state some immediate consequence of the above result which gives a manageable criterion of Mazur's property and property (X), respectively, for the space of trace class operators on some Hilbert space.

Corollary 3.18. *Let $\mathcal{H}$ be a Hilbert space. Then the space $\mathcal{T}(\mathcal{H})$ of trace class operators on $\mathcal{H}$ has Mazur's property or property (X), respectively, if and only if the Hilbert space dimension* $\dim(\mathcal{H})$ *is a non-measurable cardinal.*

Proof. The assertion follows by Theorem 3.17 and Corollary 3.9. – As for the necessity of the condition that $\dim(\mathcal{H})$ be non-measurable, we remark that it also can be obtained directly using the fact that each of the properties is hereditary and applying Edgar's result that $\ell_1(\Gamma)$ has Mazur's property or property (X), respectively, if and only if $|\Gamma|$ is non-measurable. ☐

The proof of Theorem 3.17 follows the same strategy as the one we presented for Theorem 3.12: We will reduce to the case of an L_1 space first, then decompose the latter into smaller L_1 spaces (of finite measure spaces) and apply Lemma 3.16 (ii). Here, we shall obtain property (X) for L_1 spaces of (σ-)finite measure spaces as a little corollary of the following quite general result which we think is of some interest on its own.

Theorem 3.19. *Every weakly Lindelöf determined, weakly sequentially complete direct factor of a Banach lattice has property (X).*

Corollary 3.20. *If (Ω, μ) is a σ-finite measure space, then the space $L_1(\Omega, \mu)$ has property (X).*

Proof. We know from Example 3.15 that $L_1(\Omega, \mu)$ is weakly compactly generated, hence weakly Lindelöf determined, and the remaining assertions are evident. ☐

Remark 3.21. *The following theorem is well-known (cf. [GOD 81], Thm. 7, 1), with "Erratum" concerning the separability assumption; [GOD 89], Example (2), p. 161; [EDG 83], Example (iii), p. 90): Every separable weakly sequentially complete direct factor of a Banach lattice has property (X). So we are replacing the condition of being "separable", which somehow removes all cardinality questions, by "weakly Lindelöf determined". Note that separability is a much stronger assumption, since separable spaces in particular are even weakly compactly generated.*

Finally, we stress that being weakly sequentially complete is a necessary condition to ensure property (X), as pointed out in [GOD 89], Thm. V.3 with remark thereafter on p. 159.

As noted in Lemma 3.14, weakly Lindelöf determined Banach spaces always enjoy Mazur's property. The above quite sharp result (cf. Remark 3.21) shows in a sense how much stronger property (X) is. We shall apply the known criterion for property (X) presented in Remark 3.21 (involving separability) to prove Theorem 3.19. This will be achieved by the use of a result which is of profound importance for the powerful technique of "long sequences of projections" – a by now widely used method in the theory of nonseparable Banach spaces, going back to the fundamental paper [AMI–LIN 68].

Lemma 3.22. *Let Y be a WLD Banach space, and let $(y_n) \subseteq \mathrm{Ball}(Y^*)$ be a sequence. Then there exists a projection $P : Y \to Y$ of norm 1 such that $P(Y)$ is separable and $(y_n) \subseteq P^*(Y^*)$.*

Proof. This follows from Lemma 1 in [VAL 88]. ☐

Now let us turn to Theorem 3.19.

Proof. Let Y be a Banach space satisfying the conditions of the theorem. Fix $f \in Y^{**}$ such that for every wuC series $\sum x_n$ in Y^* the equality

$$\left\langle f, w^* - \sum x_n \right\rangle = \sum \langle f, x_n \rangle$$

holds. We wish to prove that $f \in Y$. Being weakly Lindelöf determined, Y has Mazur's property by Lemma 3.14. Hence it suffices to show that for every sequence $(y_n) \subseteq \mathrm{Ball}(Y^*)$ converging w^* to 0 we have:

$$\langle f, y_n \rangle \xrightarrow{n} 0. \tag{1}$$

Fix such a sequence (y_n). Thanks to Lemma 3.22, there is a projection $P \in \mathcal{B}(Y)$ with $\|P\| = 1$, such that $P(Y)$ is separable and $(y_n) \subseteq P^*(Y^*)$. Of course,

$$P^{**}(f) \in P^{**}(Y^{**}) = P(Y)^{**}.$$

But $P(Y)$ is a *separable* weakly sequentially complete direct factor of a Banach lattice (note our assumptions about Y and the properties of P). Hence, applying the theorem stated in Remark 3.21, $P(Y)$ has property (X). Now one easily deduces from the assumptions about f that $P^{**}(f) \in P(Y)$. Since $(y_n) \subseteq P^*(Y^*)$, we finally conclude that f must satisfy condition (1). ☐

Remark 3.23. *We should mention at this point that by the same argument, using a similar result from [GOD 89] for the separable case ([GOD 89], p. 160–161), one can show the following variant of Theorem* 3.19*: Every weakly Lindelöf determined, weakly sequentially complete subspace of a Banach lattice with order continuous norm has property (X).*

Now Theorem 3.17 is only a few steps away.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$: Evident.

$(iii) \Rightarrow (iv)$: See Theorem 3.12, "$(i) \Rightarrow (ii)$".

$(iv) \Rightarrow (i)$: We proceed analogously to the proof of Theorem 3.12. – Thanks to Lemma 3.13, to prove property (X), we can assume that the von Neumann algebra $\mathcal{M}$ is abelian. Just as in the proof of Theorem 3.12, we obtain an isometric isomorphism:

$$\mathcal{M}_* = \bigoplus_{\alpha \in I}^{\ell_1} L_1(\Omega_\alpha, \mu_\alpha),$$

where $|I| \leq \kappa$ is a non-measurable cardinal and $(\Omega_\alpha, \mu_\alpha)$ are finite measure spaces. By Corollary 3.20, the spaces $L_1(\Omega_\alpha, \mu_\alpha)$ enjoy property (X). Since the latter is stable under isomorphism (cf. Remark 3.4), we conclude by applying Lemma 3.16 (ii). ☐

As an application, we state the following result which is useful in abstract harmonic analysis.

Theorem 3.24. *Let $\mathcal{G}$ be a locally compact group whose compact covering number $\mathfrak{k}(\mathcal{G})$ is non-measurable. Then $L_1(\mathcal{G})$ has property (X) (hence, in particular, Mazur's property).*

Proof. Of course, the assertion given in brackets follows from the first one; but we actually can also deduce it independently by an argument which is completely parallel to the one establishing property (X), as we shall now see.

We can assume that $\mathcal{G}$ is not σ-compact; otherwise, the proof would be finished by Corollary 3.20 or Example 3.15, respectively. Let $\mathcal{H}$ be an open σ-compact subgroup of $\mathcal{G}$. Then by Corollary 2.2 we have that $|\mathcal{G}/\mathcal{H}| = \mathfrak{k}(\mathcal{G})$, hence $|\mathcal{G}/\mathcal{H}|$ is non-measurable. Writing

$$\mathcal{G} = \dot{\bigcup_{x\mathcal{H}\in\mathcal{G}/\mathcal{H}}} x\mathcal{H},$$

we obtain an isometric isomorphism

$$L_1(\mathcal{G}) = \bigoplus_{x\mathcal{H}\in\mathcal{G}/\mathcal{H}}^{\ell_1} L_1(x\mathcal{H}),$$

where $x\mathcal{H}$ are σ-finite measure spaces (of course, the measure is the restriction of left Haar measure $\lambda_{\mathcal{G}}$ to $x\mathcal{H}$). Again by Corollary 3.20 or Example 3.15, respectively, we see that the spaces $L_1(x\mathcal{H})$, $x\mathcal{H} \in \mathcal{G}/\mathcal{H}$, all have property (X) or Mazur's property, respectively. Now Lemma 3.16 finishes the proof, where we still use the stability of the respective properties under isomorphism (cf. Remark 3.4). ☐

4 Mazur's property and property (X) of level κ

We again start with a

Definition 4.1. *Let X be a Banach space and $\kappa \geq \aleph_0$ a cardinal number.*

- *(i) A functional $f \in X^{**}$ will be called w^*-κ-continuous if for all nets $(x_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(X^*)$ of cardinality $\aleph_0 \leq |I| \leq \kappa$ with $x_\alpha \xrightarrow{w^*} 0$, we have: $\langle f, x_\alpha\rangle \longrightarrow 0$.*
- *(ii) We say that X has Mazur's property of level κ if every w^*-κ-continuous functional $f \in X^{**}$ actually is an element of X.*

Remark 4.2. *Of course, the classical property of Mazur implies Mazur's property of level $\aleph_0$. We further remark that obviously the property becomes weaker with increasing cardinality.*

One shows the following elementary facts by slightly modifying the proofs from the classical situation.

Remark 4.3. *Let $\kappa \geq \aleph_0$ be a cardinal number. Mazur's property of level κ is stable under isomorphism and passes to (closed) subspaces.*

Our aim is to prove the following

Theorem 4.4. *Let $\mathcal{G}$ be a locally compact group with compact covering number $\mathfrak{k}(\mathcal{G})$. Then $L_1(\mathcal{G})$ has Mazur's property of level $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0$.*

Remark 4.5. *If in particular the group $\mathcal{G}$ is σ-compact, the above statement follows from Example* 3.15*.*

The proof of the theorem will heavily rely on the following more general result.

Theorem 4.6. *Let I be an abstract set with $|I| \geq \aleph_0$. Let further $(X_i)_{i\in I}$ be a family of Banach spaces each having Mazur's property of level $|I|$. Then the space $\bigoplus_{i\in I}^{\ell_1} X_i$ also has this property.*

Remark 4.7. *Compare the above with Thm. 3.1 in [KAP 86] or Thm. 4.2 in [LEU 91], respectively (cf. Lemma* 3.16 *above).*

Proof. We have an isometric isomorphism:

$$\left(\bigoplus_{i\in I}^{\ell_1} X_i\right)^* = \bigoplus_{i\in I}^{\ell_\infty} X_i^*.$$

Put $\kappa := |I|$. Now let $\Phi \in \left(\bigoplus_{i\in I}^{\ell_1} X_i\right)^{**}$ be a w^*-κ-continuous functional. For each $i \in I$ denote by x_i the restriction of Φ to X_i^*. By assumption, we see that $x_i \in X_i$ for all $i \in I$. We also have that $\sum_{i\in I} \|x_i\| \leq \|\Phi\|$. Hence, the family $x := (x_i)_{i\in I}$ defines an element in $\bigoplus_{i\in I}^{\ell_1} X_i$.

We have to show that $\Phi = x$. To this end, fix $\varphi := (\varphi_i)_{i\in I} \in \bigoplus_{i\in I}^{\ell_\infty} X_i^*$. We claim that

$$\Phi(\varphi) = \sum_{i\in I} \langle \varphi_i, x_i \rangle.$$

Denote by $\mathfrak{P}_{\text{fin}}(I)$ the set of all finite subsets of I. Since $|I| \geq \aleph_0$, it follows that $\kappa = |I| = |\mathfrak{P}_{\text{fin}}(I)|$. Now for each $F \in \mathfrak{P}_{\text{fin}}(I)$, define the functional $\varphi_F \in \left(\bigoplus_{i\in I}^{\ell_1} X_i\right)^*$ through:

$$\langle \varphi_F, z \rangle := \sum_{i\in F} \langle \varphi_i, z_i \rangle \quad \text{for} \quad z = (z_i)_{i\in I} \in \bigoplus_{i\in I}^{\ell_1} X_i.$$

In other words: $\varphi_F := (\chi_F(i)\varphi_i)_{i\in I}$. Now, $(\varphi_F)_{F\in\mathfrak{P}_{\text{fin}}(I)}$ is a bounded net (directed by inclusion) which converges w^* to φ. For if $z \in \bigoplus_{i\in I}^{\ell_1} X_i$, we see that

$$\langle \varphi, z \rangle = \sum_{i\in I} \langle \varphi_i, z_i \rangle = \lim_F \sum_{i\in F} \langle \varphi_i, z_i \rangle = \lim_F \langle \varphi_F, z \rangle.$$

Hence, thanks to the w^*-κ-continuity of Φ (noting that $\kappa = |\mathfrak{P}_{\text{fin}}(I)|$), we obtain:

$$\begin{aligned}
\langle \Phi, \varphi \rangle &= \langle \Phi, w^* - \lim_F \varphi_F \rangle \\
&= \lim_F \langle \Phi, \varphi_F \rangle \\
&= \lim_F \sum_{i\in F} \langle x_i, \varphi_i \rangle \\
&= \sum_{i\in I} \langle \varphi_i, x_i \rangle,
\end{aligned}$$

which ends the proof. □

We can now deduce the assertion of Theorem 4.4.

Proof. Our procedure follows the proof of Theorem 3.24. – We can assume $\mathcal{G}$ to be non-σ-compact; otherwise Example 3.15 already yields the claim. Let $\mathcal{H}$ be an open σ-compact subgroup of $\mathcal{G}$. Then according to Corollary 2.2, we have $|\mathcal{G}/\mathcal{H}| = \mathfrak{k}(\mathcal{G}) > \aleph_0$. Writing

$$\mathcal{G} = \dot{\bigcup_{x\mathcal{H}\in\mathcal{G}/\mathcal{H}}} x\mathcal{H},$$

we have an isometric isomorphism

$$L_1(\mathcal{G}) = \bigoplus_{x\mathcal{H}\in\mathcal{G}/\mathcal{H}}^{\ell_1} L_1(x\mathcal{H})$$

with σ-finite measure spaces $x\mathcal{H}$. Again looking at Example 3.15 we see that the spaces $L_1(x\mathcal{H})$, $x\mathcal{H} \in \mathcal{G}/\mathcal{H}$, all have Mazur's property of level $\aleph_0$, so they all the more have Mazur's propety of level $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0 = \mathfrak{k}(\mathcal{G}) = |\mathcal{G}/\mathcal{H}|$. Hence by Theorem 4.6 and Remark 4.3, $L_1(\mathcal{G})$ also does. □

We finally come to a corresponding natural generalization of Godefroy-Talagrand's property (X) which has already been suggested – but never carried out – in the "Erratum" of [GOD 81] that we mentioned before. Our aim is to show that this notion of "property (X) of level κ", where now κ is an arbitrary cardinal number, only differs from the classical property (X) if κ is a measurable cardinal.

Definition 4.8. *Let X be a Banach space. A series $\sum_{\alpha\in I} f_\alpha$ in X will be called weakly unconditionally Cauchy (in short: wuC) if for every functional $\Phi \in X^*$ one has:*

$$\sum_{\alpha\in I} |\langle \Phi, f_\alpha\rangle| < \infty.$$

Remark 4.9. *One can show – cf. [SIN 70], proof of Lemma 15.1 – that a series $\sum f_\alpha$ in a dual Banach space X^* is wuC if and only if it is w^*uC (i.e., for all $x \in X$ we have $\sum_{\alpha\in I} |\langle f_\alpha, x\rangle| < \infty$).*

These preparations are sufficient for our next

Definition 4.10. *Let X be a Banach space, and $\kappa \geq \aleph_0$ a cardinal number. We say that X has property (X) of level κ if the following normality criterion holds:*

A functional $\Phi \in X^{**}$ actually belongs to X if for every wuC series $\sum_{\alpha\in I} f_\alpha$ in X^* of cardinality $|I| \leq \kappa$ one has:

$$\left\langle \Phi, w^* - \sum_{\alpha\in I} f_\alpha \right\rangle = \sum_{\alpha\in I} \langle \Phi, f_\alpha\rangle.$$

(Here, the limit $w^* - \sum_{\alpha\in I} f_\alpha$ is taken in the $\sigma(X^*, X)$-topology.)

Remark 4.11. *(i) The property clearly becomes weaker with increasing cardinality. For a very interesting – if not surprising! – connection between the classical property (X) and property (X) of higher cardinal level, see Corollary 4.15 below.*
(ii) Property (X) of level κ obviously implies Mazur's property of level κ.

We add the following statement which can easily be shown by mimicking the proof of the classical case.

Remark 4.12. *Let $\kappa \geq \aleph_0$ be a cardinal number. Property (X) of level κ is hereditary and stable under isomorphism.*

We now briefly recall the partial ordering of Banach spaces introduced by Edgar in [EDG 83]. If X and Y are Banach spaces, one defines $X \prec Y$ whenever each functional $\Phi \in X^{**}$ such that $T^{**}(\Phi) \in Y$ for all $T \in \mathcal{B}(X, Y)$, already belongs to X.

Edgar shows in [EDG 83], Prop. 10, that a Banach space X enjoys property (X) if and only if $X \prec \ell_1$. We wish to generalize this result by showing:

Proposition 4.13. *Let X be a Banach space, and $\kappa \geq \aleph_0$ a cardinal number. Then X has property (X) of level κ if and only if $X \prec \ell_1(I)$, where $|I| = \kappa$.*

Proof. We follow the argument of [EDG 83], Prop. 10; for the sake of completeness, we shall present the (short) proof.

First suppose that $X \prec \ell_1(I)$. Let $\Phi \in X^{**}$ be such that

$$\left\langle \Phi, w^* - \sum_{\alpha \in I} f_\alpha \right\rangle = \sum_{\alpha \in I} \langle \Phi, f_\alpha \rangle$$

for every w*uC series $\sum f_\alpha$ in X^*. We have to show that $\Phi \in X$. To this end, let $T \in \mathcal{B}(X, \ell_1(I))$. We denote by e_α the canonical unit vectors in $\ell_\infty(I)$. Consider the family $f_\alpha := T^*(e_\alpha)$ in X^* ($\alpha \in I$). For arbitrary $x \in X$ we have:

$$\sum_{\alpha \in I} |\langle f_\alpha, x \rangle| = \sum_{\alpha \in I} |\langle e_\alpha, Tx \rangle| = \|Tx\| < \infty.$$

We deduce that

$$\left\langle \Phi, w^* - \sum_{\alpha \in I} f_\alpha \right\rangle = \sum_{\alpha \in I} \langle \Phi, f_\alpha \rangle.$$

If $(c_\alpha)_{\alpha \in I} \subseteq \mathbb{C}$ is a bounded family, one obtains by the same argument that

$$\left\langle \Phi, w^* - \sum_{\alpha \in I} c_\alpha f_\alpha \right\rangle = \sum_{\alpha \in I} c_\alpha \langle \Phi, f_\alpha \rangle.$$

In particular, $u := (u_\alpha)_{\alpha \in I}$, where $u_\alpha := \langle \Phi, f_\alpha \rangle$, defines an element in $\ell_1(I)$. Fix an arbitrary element $g = w^* - \sum_{\alpha \in I} d_\alpha e_\alpha \in \ell_\infty(I)$. We obtain:

$$\begin{aligned} \langle T^{**}(\Phi), g \rangle &= \left\langle \Phi, w^* - \sum_{\alpha \in I} d_\alpha f_\alpha \right\rangle \\ &= \sum_{\alpha \in I} d_\alpha \langle \Phi, f_\alpha \rangle \\ &= \langle u, g \rangle. \end{aligned}$$

We thus have $T^{**}(\Phi) = u \in \ell_1(I)$, whence we deduce by our assumption that $\Phi \in X$. This establishes property (X) of level κ.

Now assume X enjoys property (X) of level κ. Let $\Phi \in X^{**}$ such that $T^{**}(\Phi) \in \ell_1(I)$ for all $T \in \mathcal{B}(X, \ell_1(I))$. Fix a w*uC series $\sum_{\alpha \in I} f_\alpha$ in X^*.

By setting $(Tx)_\alpha := f_\alpha(x)$ $(\alpha \in I)$, we define an operator $T \in \mathcal{B}(X, \ell_1(I))$. By the foregoing, we have $T^{**}(\Phi) \in \ell_1(I)$, whence:

$$\left\langle T^{**}(\Phi), w^* - \sum_{\alpha \in I} e_\alpha \right\rangle = \sum_{\alpha \in I} \langle T^{**}(\Phi), e_\alpha \rangle,$$

or in other words (noting that $T^*(e_\alpha) = f_\alpha$ for all $\alpha \in I$):

$$\left\langle \Phi, w^* - \sum_{\alpha \in I} f_\alpha \right\rangle = \sum_{\alpha \in I} \langle \Phi, f_\alpha \rangle.$$

Thanks to property (X) of level κ, we get that $\Phi \in X$. This shows $X \prec \ell_1(I)$. $\square$

Remark 4.14. *In [EDG 83], Prop. 3, it is shown that a Banach space X has Mazur's property if and only if $X \prec c_0$. We note that a characterization of Mazur's property of level κ which would correspond to Proposition 4.13 – involving the space $c_0(I)$, $|I| = \kappa$ – does not hold. Comparing the statements (1) "X has Mazur's property of level κ" and (2) "$X \prec c_0(I)$ with $|I| = \kappa$", we only have that (2) implies (1) in general.*

To see this, recall that $c_0(\Gamma) \prec c_0$ for every abstract set Γ ([EDG–ZHA 85], Prop. 1). Hence, if (2) holds, we already obtain that $X \prec c_0$ which by the above means that X has the classical property of Mazur, whence (1) holds. On the converse, if Γ is a discrete group of measurable cardinality, we deduce from Theorem 4.4 that $\ell_1(\Gamma)$ has Mazur's property of level $|\Gamma|$; but (2) would imply that $\ell_1(\Gamma) \prec c_0$, which is a contradiction to $|\Gamma|$ being measurable.

We will deduce (Corollary 4.15) from the above characterization that property (X) of level κ can only differ from the classical property (X) if κ is a measurable cardinal. So property (X) of level κ, as a criterion of normality, is interesting precisely in the hardly manageable case of "huge" non-separable Banach spaces. On the other side, excluding measurable cardinals, it is easier in some concrete case to establish property (X) of level κ for *some* cardinal number than to do so directly for the classical property (X) – just compare the technique applied in the proof of Theorem 4.4 where we showed Mazur's property of level $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0$.

Corollary 4.15. *Let $\kappa \geq \aleph_0$ be a non-measurable cardinal, X a Banach space with property (X) of level κ. Then X has the classical property (X).*

Proof. From Proposition 4.13 we know that $X \prec \ell_1(I)$, where $|I| = \kappa$. But since $|I| = \kappa$ is non-measurable, by [EDG 83], Prop. 12, it follows that $\ell_1(I) \prec \ell_1$. Transitivity of $\prec$ (cf. [EDG 83], p. 84) gives $X \prec \ell_1$. Owing to Prop. 10 in [EDG 83], this finishes the proof. $\square$

Corollary 4.16. *If $\kappa \leq \aleph_1$, then property (X) of level κ is equivalent to the classical property (X).*

By Prop. 3 in [Edg–Zha 85], for an arbitrary index set I with $|I| \geq \aleph_0$, and a family of Banach spaces $(X_\alpha)_{\alpha\in I}$ where $X_\alpha \prec \ell_1(I)$ $(\alpha \in I)$, one has:

$$\bigoplus_{\alpha\in I}^{\ell_1} X_\alpha \prec \ell_1(I).$$

Together with Proposition 4.13, this yields the following analogue of Theorem 4.6.

Theorem 4.17. *Let I be an index set with $|I| \geq \aleph_0$, and let $(X_\alpha)_{\alpha\in I}$ be a family of Banach spaces each having property (X) of level $|I|$. Then the latter is also shared by the space $\bigoplus_{\alpha\in I}^{\ell_1} X_\alpha$.*

We finish by deriving the following sharpening of Theorem 4.4.

Theorem 4.18. *Let $\mathcal{G}$ be a locally compact group with compact covering number $\mathfrak{k}(\mathcal{G})$. Then the space $L_1(\mathcal{G})$ enjoys property (X) of level $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0$.*

Proof. This follows in the same way as Theorem 4.4 by now using Corollary 3.20, Theorem 4.17 and Remark 4.12. ☐

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