2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Mean Field LQG Games with A Major Player: Continuum Parameters for Minor Players

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Abstract— **We consider a mean field LQG game model with a major player and a large number of minor players which are parametrized by a continuum set. We approximate the mean field generated by the minor players by a kernel representation using the Brownian motion of the major player, and local optimal control problems are solved for both the major player and a representative minor player via backward stochastic differential equations. The resulting set of decentralized control strategies based on consistent mean field approximations is shown to have an ε-Nash equilibrium property.**

I. INTRODUCTION

Large population stochastic dynamic games with mean field coupling have experienced intense investigation in the past decade; see, e.g., [1], [4], [10], [11], [12], [13], [16], [17], [19], [20], [21], [22], [23]. To obtain low complexity strategies, consistent mean field approximations provide a powerful approach, and in the resulting solution, each agent only needs to know its own state information and the aggregate effect of the overall population which may be pre-computed off-line. One may further establish an ε-Nash equilibrium property for the set of control strategies [12]. The technique of consistent mean field approximations is also applicable to optimization with a social objective [5], [14], [20]. The survey [3] on differential games presents a timely report of recent progresses in mean field game theory.

A naturally motivated generalization of the mean field game modeling has been introduced in [9] where a major player and a large number of minor players coexist pursuing their individual interests. Such interaction models are often seen in economic or engineering settings, simple examples being a few large corporations and many much smaller competitors, a network service provider and a large number of small users with their respective objectives. Traditionally, game models differentiating vastly different strengths of players have been well studied in cooperative game theory, and static models are usually considered [6], [7], [8].

The LQG model in [9] shows that the presence of the major player causes an interesting phenomenon called the lack of sufficient statistics. More specifically, in order to obtain asymptotic equilibrium strategies, the major player cannot simply use a strategy as a function of its current state and time; for a minor player, it cannot simply use the current states of the major player and itself. To overcome this lack of sufficient statistics for decision, the system dynamics are augmented by adding a new state, which approximates the mean field and is driven by the major player's state. This additional state enters the obtained decentralized strategy of each player and it captures the past history of the major player.

A crucial modeling assumption in [9] is that the minor players are from a finite number of classes labelled by a set $\{1,\ldots,K\}$, where players in each class share the same set of parameters in their dynamics and costs. The size of the additional state introduced in [9] depends on the number of classes so that it provides sub-mean field approximations for different classes, and this approach becomes invalid when the dynamic parameters are from an infinite set.

In this paper, we consider a population of minor players parametrized by an infinite set such as a continuum, and seek a different approach for mean field approximations. Due to the linear quadratic structure of the game with a finite number of players, it is plausible to assume that the limiting mean field is a Gaussian process and may be represented by using the driving noise of the major player. Eventually we will justify this argument by showing consistency of the mean field approximation. Given the above representation of the limiting mean field, we may approximate the original problems of the major player and a typical minor player by stochastic control problems with random coefficients in the dynamics and costs [2], [24]. This further enables the use of powerful tools from the theory of backward stochastic differential equations [2]. Also, the Gaussian property of various processes involved will play an important role and we exploit this to develop kernel representation to reduce the analysis to function spaces [18].

The organization of the paper is as follows. Section II formulates the mean field game. Section III solves two auxiliary stochastic control problems in the mean field limit. The consistency condition for mean field approximations is introduced in Section IV, and Section V shows an asymptotic Nash equilibrium property. Section VI concludes the paper.

II. THE MEAN FIELD DYNAMIC GAME MODEL

We consider the LQG game with a major player $\mathscr{A}_0$ and a population of minor players $\{\mathscr{A}_i, 1\le i\le N\}$. At time $t\ge 0$, the states of the players $\mathscr{A}_0$ and $\mathscr{A}_i$ are, respectively, denoted by $x_0(t)$ and $x_i(t)$, $1\le i\le N$. Let $(\Omega,\mathscr{F},\mathscr{F}_t,t\ge 0,P)$ be the underlying filtration. The dynamics of the $N+1$ players are given by a system of linear stochastic differential equations

$$dx_0 = (A_0x_0 + B_0u_0 + F_0x^{(N)})dt + D_0dW_0, \qquad (1)$$

$$dx_i = (A(\theta_i)x_i + B(\theta_i)u_i + F(\theta_i)x^{(N)})dt + D(\theta_i)dW_i, \qquad (2)$$

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where the initial states are given by $x_0(0)$ and $x_i(0)$, $1 \le i \le N$, and $x^{(N)} = (1/N)\sum_{i=1}^N x_i$. For simplicity, we may take $\mathscr{F}_t = \sigma\{x_i(0), W_i(s), 0 \le i \le N, s \le t\}$.

The states x_0, x_i and controls u_0, u_i are, respectively, n and n_1 dimensional vectors. The noise processes W_0, W_i are n_2 dimensional independent standard Brownian motions adapted to $\mathscr{F}_t$, which are also independent of the initial states $\{x_i(0), 0 \le i \le N\}$. The deterministic matrices $A_0, B_0, F_0, D_0, A(\cdot), B(\cdot), F(\cdot)$ and $D(\cdot)$ all have compatible dimensions. The vector θ_i is a parameter of the dynamics associated with player $\mathscr{A}_i$. We assume that θ_i takes its value from a compact subset Θ of $\mathbb{R}^d$.

The cost function for $\mathscr{A}_0$ is given by

$$J_0 = E\int_0^T \left\{\left|x_0 - \Psi_0(x^{(N)})\right|_{Q_0}^2 + u_0^T R_0 u_0\right\} dt, \qquad (3)$$

where $\Psi_0(x^{(N)}) = H_0 x^{(N)} + \eta_0$. Here and hereafter, we may write $z^T M z = |z|_M^2$ for a positive semi-definite matrix M. The cost function for $\mathscr{A}_i$, $1 \le i \le N$, is given by

$$J_i = E\int_0^T \left\{\left|x_i - \Psi(x_0, x^{(N)})\right|_{Q}^2 + u_i^T R u_i\right\} dt, \qquad (4)$$

where $\Psi(x_0, x^{(N)}) = Hx_0 + \hat{H}x^{(N)} + \eta$. The component Hx_0 in the coupling term Ψ indicates the strong influence of the major player on each minor player. In (3) and (4), all the deterministic constant matrices or vectors H_0, H, $\hat{H}$, $Q_0 \ge 0$, $Q \ge 0$, $R_0 > 0$, $R > 0$, η_0 and η have compatible dimensions.

We introduce the following assumptions:

(**A1**) The initial states $x_i(0)$, $0 \le i \le N$, are independent, $Ex_i(0) = 0$ for $1 \le i \le N$ and there is a constant C independent of N such that $\sup_{0 \le i \le N} E|x_i(0)|^2 \le C$.

(**A2**) There exists a distribution function $\mathbf{F}(\theta)$ on $\mathbb{R}^d$ such that the empirical distribution function $\mathbf{F}_N(\theta) = \frac{1}{N}\sum_{i=1}^N 1_{\{\theta_i \le \theta\}}$, where the inequality holds componentwise, converges to $\mathbf{F}$ weakly, i.e., for any bounded and continuous function $h(\theta)$ on $\mathbb{R}^d$,

$$\lim_{N\to\infty}\int_{\mathbb{R}^d} h(\theta) d\mathbf{F}_N(\theta) = \int_{\mathbb{R}^d} h(\theta) d\mathbf{F}(\theta).$$

(**A3**) $A(\cdot), B(\cdot), F(\cdot)$ and $D(\cdot)$ are continuous matrix functions of $\theta \in \Theta$, where Θ is a compact subset of $\mathbb{R}^d$.

It is worth noting that in the special case where $\Theta = \{1, 2, ..., K\}$ is a finite set, a similar game problem has been treated in [9] using the aggregation approach. By aggregating all states x_i with the same value of the parameter $\theta_i = k, 1 \le k \le K$, the mean field process $x^{(N)}(t)$ can be characterized by a Kn dimensional process $\bar{z}(t)$ described by an ordinary differential equation driven by the state of the major player. In this paper, Θ is assumed to be a compact set, not necessarily a finite set. It turns out that such a modeling gives the game a very different nature, and the Markovian state space augmentation approach developed in [9] is no longer applicable.

Remark: If a term Gx_0 appears in (2), the control perturbation of the major player will immediately impact on the mean field term and the limiting control problem of the major player will be different. For simplicity, we let (2) take the present simple form. □

III. The Limiting Control Problem

A. Two Auxiliary Optimal Control Problems

To obtain decentralized control synthesis, we formulate the auxiliary control problems within the population limit via the approximation of $x^{(N)}$ by a process z. Intuitively, due to the linear quadratic nature of the game, z should be a Gaussian process (except an additive component related to $x_0(0)$), and moreover, it should depend only on the driven noise of the major player since the noises of the minor players ought to be averaged out. Now we consider the following control problems.

Problem (I)–Optimal control of the major player. The dynamics are given by

$$\begin{cases} z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \\ dx_0 = (A_0x_0 + B_0u_0 + F_0z)dt + D_0dW_0, \end{cases} \qquad (5)$$

where z replaces $x^{(N)}$ in the finite population model. For the mean field approximation, we consider $f_1 \in C([0,T], \mathbb{R}^n)$, $f_2 \in C([0,T], \mathbb{R}^{n\times n})$, and $g \in C(\Delta, \mathbb{R}^{n\times n_2})$, where $\Delta = \{(t,s) : 0 \le s \le t \le T\}$. The cost function is given by

$$\bar{J}_0(u_0) = E\int_0^T \left\{\left|x_0 - H_0z - \eta_0\right|_{Q_0}^2 + u_0^T R_0 u_0\right\} dt. \qquad (6)$$

Problem (II)–Optimal control of the minor player. After solving problem (I), we may express the state x_0 of the major player by its initial condition and its Brownian motion, and further denote the state process by $\bar{x}_0$. By combining z, $\bar{x}_0$ with the limiting dynamics for the minor player, we introduce the equation system

$$\begin{cases} z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \\ \bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s), \\ dx_i = \big(A(\theta_i)x_i + B(\theta_i)u_i + F(\theta_i)z\big)dt + D(\theta_i)dW_i. \end{cases} \qquad (7)$$

The cost function is given by

$$\bar{J}_i(u_i) = E\int_0^T \left\{\left|x_i - H\bar{x}_0 - \hat{H}z - \eta\right|_{Q}^2 + u_i^T R u_i\right\} dt. \qquad (8)$$

B. The Analysis of Problem (I)

Lemma 1: (i) There exists a unique optimal control to problem (I) for the major player.
(ii) The pair $(\bar{x}_0, \bar{u}_0)$ is the optimal solution to problem (I) if and only if $\bar{u}_0(t) = R_0^{-1}B_0^T p_0(t)$, where $(\bar{x}_0(t), p_0(t), q_0(t))$ is the solution of the following forward-backward SDE

$$\begin{cases} d\bar{x}_0 = \big(A_0\bar{x}_0 + B_0R_0^{-1}B_0^Tp_0 + F_0z\big)dt + D_0dW_0, \\ dp_0 = \big[Q_0\big(\bar{x}_0 - H_0z - \eta_0\big) - A_0^Tp_0\big]dt + q_0dW_0, \\ \bar{x}_0(0) = x_0(0), \quad p_0(T) = 0. \end{cases} \qquad (9)$$

(iii) The forward-backward SDE (9) has a unique solution $(\bar{x}_0, p_0, q_0)$. □

Let $P_0(t) \ge 0$ be the solution of the Riccati equation

$$\begin{cases} \dot{P}_0 + P_0A_0 + A_0^TP_0 - P_0B_0R_0^{-1}B_0^TP_0 + Q_0 = 0, \\ P(T) = 0. \end{cases} \qquad (10)$$

To analyze (9), write $p_0(t) = -P_0(t)x_0(t) + \nu_0(t)$, where $\nu_0(t)$ will be determined later with the terminal condition

$\nu_0(T)=0$. Denote $\mathbb{A}_0(t)=A_0-B_0R_0^{-1}B_0^TP_0(t)$. Note that we may write $\bar{u}_0(t)=R_0^{-1}B_0^T(-P_0(t)\bar{x}_0(t)+\nu_0(t))$. By Ito's formula, it can be shown that the coupled equation system (9) is equivalent to the system of forward-backward SDE

$$\begin{cases} d\bar{x}_0=(\mathbb{A}_0\bar{x}_0+B_0R_0^{-1}B_0^T\nu_0+F_0z)dt+D_0dW_0,\\ d\nu_0=\{-\mathbb{A}_0^T\nu_0+\left[\left(P_0F_0-Q_0H_0\right)z-Q_0\eta_0\right]\}dt\\ \qquad\qquad +(q_0+P_0D_0)dW_0,\\ \bar{x}_0(0)=x_0(0), \quad \nu_0(T)=0, \end{cases} \tag{11}$$

where ν_0 is now decoupled from x_0.

To proceed, we will find a representation of $\bar{x}_0$ determined by (11) in the form

$$\bar{x}_0(t)=f_{\bar{x}_0,1}(t)+f_{\bar{x}_0,2}(t)x_0(0)+\int_0^t g_{\bar{x}_0}(t,s)dW_0(s), \tag{12}$$

where $f_{\bar{x}_0,1}\in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_0,2}\in C([0,T],\mathbb{R}^{n\times n})$, and $g_{\bar{x}_0}\in C(\Delta,\mathbb{R}^{n\times n_2})$ are to be determined.

To solve the second equation in (11), denote $\zeta_0(t)=(P_0(t)F_0-Q_0H_0)z(t)-Q_0\eta_0$ and $\mu_0(t)=q_0(t)+P_0(t)D_0$. Then by the equation of z in (5),

$$\zeta_0(t)=f_{\zeta_0,1}(t)+f_{\zeta_0,2}(t)x_0(0)+\int_0^t g_{\zeta_0}(t,s)dW_0(s),$$

where $f_{\zeta_0,1}(t)=[P_0(t)F_0-Q_0H_0]f_1(t)-Q_0\eta_0$, $f_{\zeta_0,2}(t)=[P_0(t)F_0-Q_0H_0]f_2(t)$, $g_{\zeta_0}(t,s)=[P_0(t)F_0-Q_0H_0]g(t,s)$ and

$$d\nu_0(t)=(\zeta_0(t)-\mathbb{A}_0^T(t)\nu_0(t))dt+\mu_0(t)dW_0(t).$$

Let $\Phi_0(t,s)$ be the solution of the following system

$$\begin{cases} d\Phi_0(t,s)=\mathbb{A}_0(t)\Phi_0(t,s)dt,\\ \Phi_0(s,s)=I, \qquad t\geq 0,\ s\geq 0. \end{cases} \tag{13}$$

Then by [15, Lemma A.1 (ii)],

$$\nu_0(t)=f_{\nu_0,1}(t)+f_{\nu_0,2}(t)x_0(0)+\int_0^t g_{\nu_0}(t,s)dW_0(s), \tag{14}$$

where

$$f_{\nu_0,1}(t)=\int_t^T\Phi_0^T(s_1,t)\Big[\big(Q_0H_0-P_0(s_1)F_0\big)f(s_1)+Q_0\eta_0\Big]ds_1, \tag{15}$$

$$f_{\nu_0,2}(t)=\int_t^T\Phi_0^T(s_1,t)\big(Q_0H_0-P_0(s_1)F_0\big)f(s_1)ds_1, \tag{16}$$

$$g_{\nu_0}(t,s)=\int_t^T\Phi_0^T(s_1,t)\big(Q_0H_0-P_0(s_1)F_0\big)g(s_1,s)ds_1. \tag{17}$$

We continue to solve the first equation in (11). Let $\xi_0(t)=B_0R_0^{-1}B_0^T\nu_0(t)+F_0z(t)$. Then, by the equation of z in (5) and (14)-(17),

$$\xi_0(t)=f_{\xi_0,1}(t)+f_{\xi_0,2}(t)x_0(0)+\int_0^t g_{\xi_0}(t,s)dW_0(s),$$

where $f_{\xi_0,j}(t)=B_0R_0^{-1}B_0^Tf_{\nu_0,j}(t)+F_0f_j(t)$, $j=1,2$, and $g_{\xi_0}(t,s)=B_0R_0^{-1}B_0^Tg_{\nu_0}(t,s)+F_0g(t,s)$.

We have

$$d\bar{x}_0(t)=(\xi_0(t)+M_0(t)\bar{x}_0(t))dt+D_0dW_0(t).$$

Therefore, by [15, Lemma A.1 (i)] we obtain (12), where

$$\begin{aligned} f_{\bar{x}_0,1}(t)&=\int_0^t\Phi_0(t,s_1)f_{\xi_0,1}(s_1)ds_1\\ &=\int_0^t\int_{s_1}^T\Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\\ &\quad\times\Big(\big(Q_0H_0-P_0(s_2)F_0\big)f_1(s_2)+Q_0\eta_0\Big)ds_2ds_1\\ &\qquad+\int_0^t\Phi_0(t,s_1)F_0f_1(s_1)ds_1\\ &=:[\Gamma_{0,1}f_1](t), \end{aligned} \tag{18}$$

$$\begin{aligned} f_{\bar{x}_0,2}(t)&=\Phi_0(t,0)+\int_0^t\Phi_0(t,s_1)f_{\xi_0,2}(s_1)ds_1\\ &=\int_0^t\int_{s_1}^T\Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\\ &\quad\times\big(Q_0H_0-P_0(s_2)F_0\big)f_2(s_2)ds_2ds_1\\ &\qquad+\int_0^t\Phi_0(t,s_1)F_0f_2(s_1)ds_1+\Phi_0(t,0)\\ &=:[\Gamma_{0,2}f_2](t), \end{aligned} \tag{19}$$

$$\begin{aligned} g_{\bar{x}_0}(t,s)&=\int_s^t\Phi_0(t,s_1)g_{\xi_0}(s_1,s)ds_1+\Phi_0(t,s)D_0\\ &=\int_s^t\int_{s_1}^T\Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\\ &\quad\times\big(Q_0H_0-P_0(s_2)F_0\big)g(s_2,s)ds_2ds_1\\ &\qquad+\int_s^t\Phi_0(t,s_1)F_0g(s_1,s)ds_1+\Phi_0(t,s)D_0\\ &=:[\Lambda_0g](t,s). \end{aligned} \tag{20}$$

C. The Analysis of Problem (II)

Lemma 2: (i) There exists a unique optimal control to problem (II).
(ii) The pair $(\bar{x}_i,\bar{u}_i)$ is the optimal solution to problem (II) if and only if $\bar{u}_i(t)=R^{-1}B^T(\theta_i)p_i(t)$, where $(\bar{x}_i(t),p_i(t),q_i(t),r_i(t))$ is the solution of the forward-backward SDE

$$\begin{cases} d\bar{x}_i=\Big(A(\theta_i)\bar{x}_i+B(\theta_i)R^{-1}B^T(\theta_i)p_i+F(\theta_i)z\Big)dt\\ \qquad\qquad +D(\theta_i)dW_i,\\ dp_i=\Big[Q\big(\bar{x}_i-H\bar{x}_0-\hat{H}z-\eta\big)-A^T(\theta_i)p_i\Big]dt\\ \qquad\qquad +q_idW_i+r_idW_0,\\ \bar{x}_i(0)=x_i(0), \quad p_i(T)=0. \end{cases} \tag{21}$$

(iii) The forward-backward SDE (21) has a unique solution $(\bar{x}_i,p_i,q_i,r_i)$. □

Let $P_{\theta_i}(t)\geq 0$ be the solution of the Riccati equation

$$\begin{cases} \dot{P}_{\theta_i}+P_{\theta_i}A(\theta_i)+A^T(\theta_i)P_{\theta_i}\\ \qquad -P_{\theta_i}B(\theta_i)R_0^{-1}B^T(\theta_i)P_{\theta_i}+Q=0,\\ P_{\theta_i}(T)=0. \end{cases} \tag{22}$$

Write $p_i(t)=-P_{\theta_i}(t)x_i(t)+\nu_{\theta_i}(t)$, where $\nu_{\theta_i}(t)$ will be determined later satisfying the terminal condition $\nu_{\theta_i}(T)=0$.

Denote $\mathbb{A}_{\theta_i}(t)=A(\theta_i)-B(\theta_i)R^{-1}B^T(\theta_i)P_{\theta_i}(t)$. Similar to (11), the coupled equation system (21) is equivalent to the following forward-backward SDE

$$\begin{cases} d\bar{x}_i=\Big\{\mathbb{A}_{\theta_i}\bar{x}_i+B(\theta_i)R^{-1}B^T(\theta_i)\nu_{\theta_i}+F(\theta_i)z\Big\}dt\\ \qquad\quad +D(\theta_i)dW_i,\\ d\nu_{\theta_i}=\Big\{-\mathbb{A}^T_{\theta_i}\nu_{\theta_i}+(P_{\theta_i}F(\theta_i)-Q\hat{H})z-QH\bar{x}_0-Q\eta\Big\}dt\\ \qquad\quad +(q_i+P_{\theta_i}D(\theta_i))dW_i+r_idW_0,\\ \bar{x}_i(0)=x_i(0),\quad \nu_{\theta_i}(T)=0. \end{cases} \tag{23}$$

We will represent $\bar{x}_i(t)$ in the form

$$\begin{aligned}\bar{x}_i(t)=&f_{\bar{x}_i,1}(t)+f_{\bar{x}_i,2}(t)x_0(0)+f_{\bar{x}_i,3}(t)x_i(0)\\ &+\int_0^t g_{\bar{x}_i}(t,s)dW_0(s)+\int_0^t h_{\bar{x}_i}(t,s)dW_i(s),\end{aligned} \tag{24}$$

where $f_{\bar{x}_i,1}\in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_i,2},f_{\bar{x}_i,3}\in C([0,T],\mathbb{R}^{n\times n})$, and $g_{\bar{x}_i},h_{\bar{x}_i}\in C([0,T],\mathbb{R}^{n\times n_2})$ are to be determined.

Let $\zeta_i(t)=(P_{\theta_i}(t)F(\theta_i)-Q\hat{H})z(t)-QH\bar{x}_0(t)-Q\eta$, $\lambda_i(t)=q_i(t)+P_{\theta_i}(t)D(\theta_i)$. Then by (12),

$$\zeta_i(t)=f_{\zeta_i,1}(t)+f_{\zeta_i,2}(t)x_0(0)+\int_0^t g_{\zeta_i}(t,s)dW_0(s),$$

where

$$\begin{aligned} f_{\zeta_i,1}(t)&=\big[P_{\theta_i}(t)F(\theta_i)-Q\hat{H}\big]f_1(t)-QHf_{\bar{x}_0,1}(t)-Q\eta,\\ f_{\zeta_i,2}(t)&=\big[P_{\theta_i}(t)F(\theta_i)-Q\hat{H}\big]f_2(t)-QHf_{\bar{x}_0,2}(t),\\ g_{\zeta_i}(t,s)&=\big[P_{\theta_i}(t)F(\theta_i)-Q\hat{H}\big]g(t,s)-QHg_{\bar{x}_0}(t,s).\end{aligned}$$

By (23), we have

$$d\nu_{\theta_i}(t)=[\zeta_i(t)-\mathbb{A}^T_{\theta_i}\nu_{\theta_i}(t)]dt+r_i(t)dW_0(t)+\lambda_i(t)dW_i(t).$$

Let $\Phi_{\theta_i}(t,s)$ be the solution of

$$\begin{cases} d\Phi_{\theta_i}(t,s)=\mathbb{A}_{\theta_i}(t)\Phi_{\theta_i}(t,s)dt,\\ \Phi_{\theta_i}(s,s)=I,\qquad t,s\ge 0.\end{cases} \tag{25}$$

Then by [15, Lemma A.2 (ii)],

$$\nu_{\theta_i}(t)=f_{\nu_{\theta_i},1}(t)+f_{\nu_{\theta_i},2}(t)x_0(0)+\int_0^t g_{\nu_{\theta_i}}(t,s)dW_0(s), \tag{26}$$

where

$$\begin{aligned} f_{\nu_{\theta_i},1}(t)=&\int_t^T\Phi^T_{\theta_i}(s_1,t)\Big[\big(Q\hat{H}-P_{\theta_i}(s_1)F(\theta_i)\big)f_1(s_1)\\ &+QHf_{\bar{x}_0,1}(s_1)+Q\eta\Big]ds_1,\end{aligned} \tag{27}$$

$$\begin{aligned} f_{\nu_{\theta_i},2}(t)=&\int_t^T\Phi^T_{\theta_i}(s_1,t)\Big[\big(Q\hat{H}-P_{\theta_i}(s_1)F(\theta_i)\big)f_2(s_1)\\ &+QHf_{\bar{x}_0,2}(s_1)\Big]ds_1,\end{aligned} \tag{28}$$

$$\begin{aligned} g_{\nu_{\theta_i}}(t,s)=&\int_t^T\Phi^T_{\theta_i}(s_1,t)\Big[\big(Q\hat{H}-P_{\theta_i}(s_1)F(\theta_i)\big)g(s_1,s)\\ &+QHg_{\bar{x}_0}(s_1,s)\Big]ds_1.\end{aligned} \tag{29}$$

Next, let $\xi_i(t)=B(\theta_i)R^{-1}B^T(\theta_i)\nu_{\theta_i}(t)+F(\theta_i)z(t)$. Then by the equation of z in (5) and (26), $\xi_i(t)=f_{\xi_i,1}(t)+f_{\xi_i,2}(t)x_0(0)+\int_0^t g_{\xi_i}(t,s)dW_0(s)$ with

$$f_{\xi_i,j}(t)=B(\theta_i)R^{-1}B^T(\theta_i)f_{\nu_{\theta_i},j}(t)+F(\theta_i)f_j(t),\ j=1,2, \tag{30}$$

$$g_{\xi_i}(t,s)=B(\theta_i)R^{-1}B^T(\theta_i)g_{\nu_{\theta_i}}(t,s)+F(\theta_i)g(t,s). \tag{31}$$

We have

$$d\bar{x}_i(t)=\big(\xi_i(t)+\mathbb{A}_{\theta_i}(t)\bar{x}_i(t)\big)dt+D(\theta_i)dW_i(t).$$

Therefore from [15, Lemma A.2 (i)], we obtain (24), where

$$\begin{aligned} f_{\bar{x}_i,1}(t)&=\int_0^t\Phi_{\theta_i}(t,s_1)f_{\xi_i,1}(s_1)ds_1\\ &=\int_0^t\Phi_{\theta_i}(t,s_1)B(\theta_i)R^{-1}B^T(\theta_i)\int_{s_1}^T\Phi^T_{\theta_i}(s_2,s_1)\times\\ &\quad\times\Big[\big(Q\hat{H}-P_{\theta_i}(s_2)F(\theta_i)\big)f_1(s_2)+QHf_{\bar{x}_0,1}(s_2)+Q\eta\Big]ds_2ds_1\\ &\quad+\int_0^t\Phi_{\theta_i}(t,s_1)F(\theta_i)f_1(s_1)ds_1\\ &=:[\Gamma_{\theta_i,1}f_1](t),\end{aligned} \tag{32}$$

$$\begin{aligned} f_{\bar{x}_i,2}(t)&=\int_0^t\Phi_{\theta_i}(t,s_1)f_{\xi_i,2}(s_1)ds_1\\ &=\int_0^t\Phi_{\theta_i}(t,s_1)B(\theta_i)R^{-1}B^T(\theta_i)\int_{s_1}^T\Phi^T_{\theta_i}(s_2,s_1)\times\\ &\quad\times\Big[\big(Q\hat{H}-P_{\theta_i}(s_2)F(\theta_i)\big)f_2(s_2)+QHf_{\bar{x}_0,2}(s_2)\Big]ds_2ds_1\\ &\quad+\int_0^t\Phi_{\theta_i}(t,s_1)F(\theta_i)f_2(s_1)ds_1\\ &=:[\Gamma_{\theta_i,2}f_2](t),\end{aligned} \tag{33}$$

and

$$\begin{aligned} g_{\bar{x}_i}(t,s)&=\int_s^t\Phi_{\theta_i}(t,s_1)g_{\xi_i}(s_1,s)ds_1\\ &=\int_s^t\Phi_{\theta_i}(t,s_1)B(\theta_i)R^{-1}B^T(\theta_i)\int_{s_1}^T\Phi^T_{\theta_i}(s_2,s_1)\times\\ &\quad\times\Big[\big(Q\hat{H}-P_{\theta_i}(s_2)F(\theta_i)\big)g(s_2,s)+QHg_{\bar{x}_0}(s_2,s)\Big]ds_2ds_1\\ &\quad+\int_s^t\Phi_{\theta_i}(t,s_1)F(\theta_i)g(s_1,s)ds_1\\ &=:[\Lambda_{\theta_i}g](t,s),\end{aligned} \tag{34}$$

and furthermore,

$$f_{\bar{x}_i,3}(t)=\Phi_{\theta_i}(t,0),\quad h_{\bar{x}_i}(t,s)=\Phi_{\theta_i}(t,s)D(\theta_i). \tag{35}$$

The importance of (24) is that it explicitly relates the functions in the limiting mean field to the representation of $\bar{x}_i$, and hence in the analysis of the replica of the mean field z, we may solely focus on function spaces.

IV. The Consistency Condition

We now introduce the consistency condition for the mean field approximation. More precisely, when the controls obtained in Section III are applied, the mean field replicated by the closed loop in the population limit should coincide with the one assumed at the beginning. To proceed, denote

$$[\Gamma_j f](t)=\int_\Theta [\Gamma_{\theta,j} f_j](t)d\mathbf{F}(\theta),\quad 0\le t\le T,\ j=1,2, \tag{36}$$

$$[\Lambda g](t,s)=\int_\Theta [\Lambda_\theta g](t,s)d\mathbf{F}(\theta),\quad 0\le s\le t\le T, \tag{37}$$

for $f_1\in C([0,T],\mathbb{R}^n)$, $f_2\in C([0,T],\mathbb{R}^{n\times n})$ and $g\in C(\Delta,\mathbb{R}^{n\times n_2})$, where $\Delta=\{(t,s): 0\le s\le t\le T\}$. Here, $[\Gamma_{\theta,j}f_j](t)$, $j=1,2$, and $[\Lambda_\theta g](t,s)$ are, respectively, defined as in (32)-(34) with $\theta_i=\theta$.

Lemma 3: Assume (**A1**)-(**A3**).

(i) Γ_1 is a mapping from $C([0,T],\mathbb{R}^n)$ to $C([0,T],\mathbb{R}^n)$.

(ii) Γ_2 is a mapping from $C([0,T],\mathbb{R}^{n\times n})$ to $C([0,T],\mathbb{R}^{n\times n})$.

(iii) Λ is a mapping from $C(\Delta,\mathbb{R}^{n\times n_2})$ to $C(\Delta,\mathbb{R}^{n\times n_2})$. □

Denote

$$C_{NCE}=C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^{n\times n})\times C(\Delta,\mathbb{R}^{n\times n_2}).$$

Definition 4: A triple $(f_1,f_2,g)\in C_{NCE}$ is called a consistent solution to the Nash certainty equivalence (NCE) equation system if

$$\begin{cases} f_j(t)=[\Gamma_j f_j](t), & 0\le t\le T,\ j=1,2,\\ g(t,s)=[\Lambda g](t,s), & 0\le s\le t\le T. \end{cases} \tag{38}$$

□

Denote the linear operators $\bar\Gamma_{0,1}$, $\bar\Gamma_{0,2}$ and $\bar\Lambda_0$ on $C([0,1],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$ and $C(\Delta,\mathbb{R}^{n\times n_2})$, respectively, as follows:

$$\begin{aligned}[\bar\Gamma_{0,j}f_j](t)=&\int_0^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\\ &\times\big(Q_0H_0-P_0(s_2)F_0\big)f_j(s_2)ds_2ds_1\\ &+\int_0^t\Phi_0(t,s_1)F_0f_j(s_1)ds_1,\ j=1,2,\\ [\bar\Lambda_0 g](t,s)=&\int_s^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\\ &\times\big(Q_0H_0-P_0(s_2)F_0\big)g(s_2,s)ds_2ds_1\\ &+\int_s^t\Phi_0(t,s_1)F_0g(s_1,s)ds_1,\end{aligned}$$

which are obtained by retaining the linear term of the affine operators $\Gamma_{0,j}$ and Λ_0, respectively.

Corresponding to $\Gamma_{\theta,1}$, $\Gamma_{\theta,2}$ and $\Lambda_\theta g$, define the linear operators $\bar\Gamma_1$, $\bar\Gamma_2$, and $\bar\Lambda$ on $C([0,1],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$ and $C(\Delta,\mathbb{R}^{n\times n_2})$, respectively, as follows:

$$\begin{aligned}[\bar\Gamma_{\theta,j}f_j](t)=&\int_0^t\int_{s_1}^T\Phi_\theta(t,s_1)B(\theta)R^{-1}B^T(\theta)\Phi_\theta^T(s_2,s_1)\times\\ &\times\Big[\big(Q\hat H-P_\theta(s_2)F(\theta)\big)f_j(s_2)+QH[\bar\Gamma_{0,j}f_j](s_2)\Big]ds_2ds_1\\ &+\int_0^t\Phi_\theta(t,s_1)F(\theta)f_j(s_1)ds_1,\quad j=1,2,\\ [\bar\Lambda_\theta g](t,s)=&\int_s^t\int_{s_1}^T\Phi_\theta(t,s_1)B(\theta)R^{-1}B^T(\theta)\Phi_\theta^T(s_2,s_1)\times\\ &\times\Big[\big(Q\hat H-P_\theta(s_2)F(\theta)\big)g(s_2,s)+QH[\bar\Lambda_0 g](s_2,s)\Big]ds_2ds_1\\ &+\int_s^t\Phi_\theta(t,s_1)F(\theta)g(s_1,s)ds_1.\end{aligned}$$

Let $C([0,1],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$ and $C(\Delta,\mathbb{R}^{n\times n_2})$ be endowed with the usual sup-norms $\|\cdot\|_\infty$ so that they are all Banach spaces. Define the linear operators

$$\begin{aligned}[\bar\Gamma_j f_j](t)&=\int_\Theta[\bar\Gamma_{\theta,j}f_j](t)d\mathbf{F}(\theta),\quad j=1,2,\\ [\bar\Lambda g](t,s)&=\int_\Theta[\bar\Lambda_\theta g](t,s)d\mathbf{F}(\theta).\end{aligned}$$

For $f_2\in C([0,T],\mathbb{R}^{n\times n})$, we write $f_2=[f_{2,1},\dots,f_{2,n}]$, where $f_{2,i}\in C([0,T],\mathbb{R}^n)$ for each $i=1,\dots,n$. Then $[\bar\Gamma_2 f_2](t)=\big[[\bar\Gamma_1 f_{2,1}](t),\dots,[\bar\Gamma_1 f_{2,n}](t)\big]$. We have the following result for the NCE equation system.

Theorem 5: Under (**A1**)-(**A3**), if the norms of $\bar\Gamma$ and $\bar\Lambda$ satisfy $\|\bar\Gamma_1\|<1$ and $\|\bar\Lambda\|<1$, then there exists a unique solution $(f_1,f_2,g)\in C_{NCE}$ to (38). □

V. Asymptotic Equilibrium Analysis

Throughout this section we assume that there exists a solution $(f_1,f_2,g)\in C_{NCE}$ to the NCE equation system (38). Define $\varepsilon_N\ge 0$ by $\varepsilon_N^2=\varepsilon_{f_1,N}^2+\varepsilon_{f_2,N}^2+\varepsilon_{g,N}^2$, where

$$\varepsilon_{f_j,N}^2=\int_0^T\Big|\int_\Theta[\Gamma_{\theta,j}f_j](t)d\mathbf{F}_N(\theta)-\int_\Theta[\Gamma_{\theta,j}f_j](t)d\mathbf{F}(\theta)\Big|^2dt,$$

$$\varepsilon_{g,N}^2=\int_0^T\int_0^t\Big|\int_\Theta[\Lambda_\theta g](t,s)d\mathbf{F}_N(\theta)-\int_\Theta[\Lambda_\theta g](t,s)d\mathbf{F}(\theta)\Big|^2dsdt.$$

Lemma 6: Under (**A1**)-(**A3**), $\lim_{N\to\infty}\varepsilon_N=0$. □

Consider the system (1)-(2). Let the control laws of $\mathscr{A}_0$ and $\mathscr{A}_i$, $1\le i\le N$, be given by

$$\hat u_0(t)=R_0^{-1}B_0^T\big(-P_0(t)x_0(t)+\nu_0(t)\big), \tag{39}$$

$$\hat u_i(t)=R^{-1}B^T(\theta_i)\big(-P_{\theta_i}(t)x_i(t)+\nu_{\theta_i}(t)\big), \tag{40}$$

where $\nu_0(t)$ and $\nu_{\theta_i}(t)$ are determined by (9) and (21) corresponding to the solution (f_1,f_2,g) to (38). Their explicit solutions are given by (14) and (26). After the control laws (39)-(40) are applied, the dynamics of $\mathscr{A}_0$ and $\mathscr{A}_i$ may be written in the form

$$dx_0=\Big\{\mathbb{A}_0x_0+B_0R_0^{-1}B_0^TP_0\nu_0+F_0x^{(N)}\Big\}dt+D_0dW_0, \tag{41}$$

$$\begin{aligned}dx_i=&\Big\{\mathbb{A}_{\theta_i}x_i+B(\theta_i)R^{-1}B^T(\theta_i)P_{\theta_i}\nu_{\theta_i}+F(\theta_i)x^{(N)}\Big\}dt\\ &+D(\theta_i)dW_i,\quad 1\le i\le N,\end{aligned} \tag{42}$$

where $x^{(N)}=(1/N)\sum_{i=1}^N x_i$.

We now construct the limiting equation system for the $N+1$ players

$$d\bar x_0=\Big\{\mathbb{A}_0\bar x_0+B_0R_0^{-1}B_0^TP_0\nu_0+F_0z\Big\}dt+D_0dW_0, \tag{43}$$

$$\begin{aligned}d\bar x_i=&\Big\{\mathbb{A}_{\theta_i}\bar x_i+B(\theta_i)R^{-1}B^T(\theta_i)P_{\theta_i}\nu_{\theta_i}+F(\theta_i)z\Big\}dt\\ &+D(\theta_i)dW_i,\quad 1\le i\le N,\end{aligned} \tag{44}$$

with the initial conditions $\bar x_i(0)=x_i(0)$. We have the proposition on the mean field approximation.

Proposition 7: Assume (**A1**)-(**A3**). Then

$$E\int_0^T\big|z(t)-\bar x^{(N)}(t)\big|^2dt=O\Big(\varepsilon_N^2+1/N\Big),$$

where $\bar x^{(N)}=(1/N)\sum_{i=1}^N\bar x_i$. □

By Proposition 7 we may further establish the next theorem.

Theorem 8: Assume (**A1**)-(**A3**). We have

$$E\int_0^T \Big(\big|z(t)-x^{(N)}(t)\big|^2+\sup_{0\le j\le N}\big|x_j(t)-\bar{x}_j(t)\big|^2\Big)dt$$
$$=O\Big(\varepsilon_N^2+1/N\Big). \tag{45}$$

□

Consider the system of $N+1$ agents described by (1) and (2). Let the class $\mathscr{U}_W$ consist of all processes y_W of the form

$$y_W(t)=\int_0^t [h_0(t,s)dW_0(s),\dots,h_N(t,s)dW_N(s)]^T$$

where each h_j is an $\mathbb{R}^{n\times n_2}$-valued bounded measurable function on Δ.

For any $i=0,...,N$, the admissible control $\mathscr{U}_i$ of agent $\mathscr{A}_i$ consists of control u_i as a time dependent function linear in $x_0,x_1,\dots,x_N,y_W$ for some $y_W\in\mathscr{U}_W$. The resulting control of a player may not be purely in a feedback form since the noise process may be used via y_W; this more general form of controls is necessary in order to include the decentralized controls (39)-(40) that we have derived. Since the control still uses the players' states, $(u_0,u_1,\dots,u_N)$ is in a partial feedback form, and will be called a set of partial feedback strategies. Note that $\mathscr{U}_i$ is not restricted to be decentralized. Given each set of strategies in $\mathscr{U}_0\times\ldots\times\mathscr{U}_N$, the closed-loop system has a unique strong solution. For $i=0,...,N$, denote $u_{-i}=(u_0,u_1,...,u_{i-1},u_{i+1},...,u_N)$.

Definition 9: A set of controls $u_i\in\mathscr{U}_i$, $0\le i\le N$, for the $N+1$ players is called an ε-Nash equilibrium with respect to the costs J_i, $0\le i\le N$, where $\varepsilon\ge 0$, if for any i, $0\le i\le N$, we have $J_i(u_i,u_{-i})\le J_i(u_i',u_{-i})+\varepsilon$, when any alternative u_i' is applied by player $\mathscr{A}_i$. □

Theorem 10: Assume (**A1**)-(**A3**). Let $\bar{u}_0$ and $\bar{u}_i$ be the optimal controls in the limiting control problems (I) and (II). For $0\le j\le N$,

$$\Big|J_j(\hat{u}_j,\hat{u}_{-j})-\bar{J}_j(\bar{u}_j)\Big|=O\Big(\varepsilon_N+1/\sqrt{N}\Big).$$

□

By using Theorem 10 we can further establish the next theorem.

Theorem 11: Assume (**A1**)-(**A3**). Then the set of controls $\hat{u}_j$, $0\le j\le N$, for the $N+1$ players is an ε-Nash equilibrium, i.e., for $0\le j\le N$,

$$J_j(\hat{u}_j,\hat{u}_{-j})-\varepsilon\le\inf_{u_j}J_j(u_j,\hat{u}_{-j})\le J_i(\hat{u}_j,\hat{u}_{-j}),$$

where $0\le\varepsilon=O(\varepsilon_N+1/\sqrt{N})$. □

VI. Conclusion

This paper considers mean field LQG games with a major player and a continuum-parametrized minor players. The mean field structure does not allow the Markovian state space augmentation approach developed in the previous work [9]. We introduce random Gaussian mean field approximations and solve the resulting limiting problems as stochastic optimal control with random coefficients, and we further derive decentralized controls for the players.

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