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Para Fabián

Para Luisito y Tomasín

Preface

Profinite groups are Galois groups, which we view as topological groups. In this book the theory of profinite graphs is developed as a natural tool in the study of some aspects of profinite and abstract groups. Our approach is modelled on the by now classical Bass–Serre theory of abstract groups acting on abstract trees as it appears in J.-P. Serre's monograph 'Trees'.

We think of a graph Γ as the union of its sets of vertices V and edges E. A graph Γ is profinite if it is endowed with a profinite topology (i.e., a compact, Hausdorff and totally disconnected topology), in such a way that the functions defining the origin and terminal points are continuous. A natural example of a profinite graph is the Cayley graph $\Gamma(G, X)$ of a profinite group G with respect to a closed subset X, say finite, of G: the vertices of Γ are the elements of G, and its directed edges have the form (g, x) $(g \in G, x \in X)$ with origin $d_0(g, x) = g$ and terminal $d_1(g, x) = gx$. Then the topology of G naturally induces a profinite topology on $\Gamma(G, X)$.

Part I of this book contains an exposition of the theory of profinite graphs and how it relates to and is motivated by the theory of profinite groups. Part II deals with applications to profinite groups, while Part III is dedicated to the study of certain properties of abstract groups with the help of tools developed in Parts I and II.

Our aim in Parts I and II has been to make the exposition self-contained, and familiarity with the theory of abstract graphs and groups is not strictly necessary. However, knowledge of the Bass–Serre theory certainly helps, and throughout these two parts we often indicate the interconnections. These connections are in fact the main tools for some of the applications to abstract groups in Part III, where results and ideas ranging from topology and abstract group theory to automata theory are used freely.

One fundamental difference with the abstract case is that a profinite group acting freely on a profinite tree need not be a free profinite group (it is just projective). This leads to a study of Galois coverings of profinite graphs and fundamental groups of profinite graphs. Throughout the book we have tried to be as general as reasonably possible, and so we consider pro-$\mathcal{C}$ groups, where $\mathcal{C}$ is a class of finite groups, rather than profinite groups in general. Consequently the book includes studies of Galois $\mathcal{C}$-coverings, $\mathcal{C}$-trees, fundamental groups of graphs of pro-$\mathcal{C}$ groups, etc.

Part I (Chaps. 2–6) includes the development of free products of pro-$\mathcal{C}$ groups continuously indexed by a topological profinite space, and a full treatment of the fundamental pro-$\mathcal{C}$ group of a graph of pro-$\mathcal{C}$ groups.

Part II (Chaps. 7–10) contains applications to the structure of profinite groups. In Chap. 7 we describe subgroups of fundamental groups of graphs of profinite groups; in particular, an analogue of the Kurosh subgroup theorem for open subgroups of free products of pro-$\mathcal{C}$ groups is established. Chapter 8 describes the properties of minimal G-invariant subtrees of a tree on which the group G acts; this is done for profinite as well as abstract groups and graphs. The study of such minimal trees was initiated by Tits when G is cyclic and acts without fixed points on an abstract tree. It turns out that the connections between these types of minimal subtrees in the abstract and profinite cases provides a powerful tool to study certain properties in abstract groups. Chapters 9 and 10 of Part II deal mainly with homology. Chapter 9 includes a theorem of Neukirch and a generalization of Mel'nikov characterizing homologically when a profinite group is the free product of a collection of subgroups continuously indexed by a topological (profinite) space; this plays the role of the usual combinatorial description of free products in the case of abstract groups. This chapter also contains a Kurosh-like theorem for countably generated closed subgroups of free products of pro-p groups due to D. Haran and O. Mel'nikov independently. Chapter 10 includes the well-known theorem of J.-P. Serre that asserts that a torsion-free pro-p group G with an open free pro-p subgroup must be free pro-p. There is also a generalization of this result due to C. Scheiderer, where one allows torsion in G. Using this, the chapter also contains a study of the subgroup of fixed points of an automorphism of a free pro-p group.

Part III (Chaps. 11–15) contains applications to abstract groups. These include generalizations of a theorem of Marshall Hall that asserts that a finitely generated subgroup H of an abstract free group Φ is the intersection of the subgroups of finite index in Φ that contain H; an algorithm to compute the closure of a finitely generated subgroup H of an abstract free group Φ in the pro-p topology of Φ; and applications to the theory of formal languages and finite monoids. Also included is the study of certain properties that hold for an abstract group if and only if they hold for the finite quotients of that group, e.g., conjugacy separability for an abstract group R: for $x, y \in R$, these elements are conjugate in R if their images are conjugate in every finite quotient group of R.

The book ranges over a large number of areas and results, but we have not intended to make this into an encyclopedia of the subject. Part I gives a fairly complete account of profinite graphs and their connection with profinite groups. However in Part II and, even more, in Part III, I have made a choice of topics to illustrate some results and methods. At the end of each of the three parts of the book there is a section with historical comments on the development of the fundamental ideas and theorems, statements of additional results, references to related topics, and open questions.

In an effort to make the book self-contained, the first chapter includes a review of basic notions and results about profinite spaces, profinite groups and homology that are used frequently throughout the monograph. Appendix A deals with aspects of abstract graphs that are of interest in the book. The main purpose has been to develop a terminology common to abstract and profinite graphs. Appendix B contains a proof of a theorem of M. Benois about rational languages in free abstract groups.

I have been indebted to many colleagues during the writing of this book. Throughout the years I have had many mathematical discussions with my longtime collaborator Pavel Zalesskii that have helped to clarify some topics developed here; it is a pleasure to acknowledge with thanks my debt to him. I thank John Dixon, Wolfgang Herfort, Dan Segal and Benjaming Steinberg, who have read parts of the manuscript and have made very useful comments, corrections and suggestions. Jean-Eric Pin has provided helpful references, and I am very grateful to him for this.

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Madrid–Ottawa Luis Ribes
March, 2017

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