Name (PRINT): _____ Student # _____

Test #2
Mathematics 69.107
Oct 20, 2000
Full marks: 40

Answer all questions. Use non-programmable calculator if needed.

(1) [12 marks] Evaluate the following limits:

(a) [4 marks] $\lim_{x\to 0} \frac{\tan(x)}{x+\sin(x)}$

$$\lim_{x\to 0} \frac{\tan(x)}{x+\sin(x)} = \lim_{x\to 0} \frac{\sec^2(x)}{1+\cos(x)} = \frac{1}{1+1} = \frac{1}{2}.$$

(b) [4 marks] $\lim_{x\to\infty} e^{-x}\ln(x)$

$$\lim_{x\to\infty} e^{-x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0.$$

(c) [4 marks] $\lim_{x\to 0} \frac{\sin^{-1}(x)}{\tan^{-1}(x)}$

$$\lim_{x\to 0} \frac{\sin^{-1}(x)}{\tan^{-1}(x)} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{1+x^2}} = \lim_{x\to 0} \frac{1+x^2}{\sqrt{1-x^2}} = 1.$$

(2) [10 marks]

The radioactive element carbon-14 has a half-life of 5750 years. (This means that after 5750 years half of the original amount has decayed.) If 100 gms. of this element are present initially, how much will be left after 1000 years?

Let $y(t)$ be the amount left after t years. We know that $y(0) = 100$ gms. and $y(5750) = 50$ gms. Then, we have that

$$y(t) = y(0)e^{kt} = 100e^{kt}. \tag{1}$$

Using Equation (1), we have $50 = y(5750) = 100e^{5750k}$. Therefore, $e^{5750k} = \frac{1}{2}$, implying $5750k = \ln 1 - \ln 2$, and $k = -\frac{\ln 2}{5750}$. Substituting back in Equation (1):

$$\begin{aligned} y(t) &= (100)e^{-\frac{\ln 2}{5750}t} = (100)e^{-\frac{t}{5750}\ln 2} \\ &= (100)e^{\ln 2^{-\frac{t}{5750}}} = (100)2^{-\frac{t}{5750}}. \end{aligned}$$

Finally,

$$y(1000) = (100)2^{-\frac{1000}{5750}} = (100)2^{-\frac{4}{23}} = 88.6 \text{ gms}.$$

(3) [18 marks] Let $f(x) = \frac{e^x}{x^2}$.

(a) [2 marks] Find all the critical points of $f(x)$.

$$f'(x) = \frac{x^2e^x - 2xe^x}{x^4} = \frac{(x-2)e^x}{x^3}.$$

Thus, $x = 2$ is the only critical point (note that $x = 0$ is not in the domain of the function).

(b) [3 marks] Find the intervals where the function is increasing and decreasing.

Intervals	$x-2$	x^3	$f'(x)$	$f(x)$
$-\infty < x < 0$	$-$	$-$	$+$	$\nearrow$
$0 < x < 2$	$-$	$+$	$-$	$\searrow$
$2 < x < \infty$	$+$	$+$	$+$	$\nearrow$.

Thus, f is increasing on $(-\infty, 0)$ and $(2, \infty)$; f is decreasing on $(0, 2)$.

(c) [2 marks] Find any local maximum and minimum, and values of such maximum and minimum.

$f'(x) < 0$ to the left of 2, and $f'(x) > 0$ to the right of 2, thus there is a local minimum at $x = 2$. The local minimum value is $f(2) = \frac{1}{4}e^2$.

(d) [3 marks] Find any vertical asymptotes (evaluate the appropriate limits).

The line $x = 0$ is a vertical asymptote since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x}{x^2} = \infty.$$

(e) [2 marks] Find where the function is concave up or concave down and inflection points, if any.

$$f''(x) = \frac{(1+x-2)x^3e^x - (x-2)e^x 3x^2}{x^6} = \frac{(x^2-4x+6)e^x}{x^4}.$$

The second derivative is positive for all x since $x^2 - 4x + 6$ has no real roots. Thus, the function is concave up and there are no inflection points.

(f) [3 marks] Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and hence the horizontal asymptote(s).

First we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} == \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Second, we have

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{e^x}{x^2} = 0.$$

The last equation implies that the line $y = 0$ is a horizontal asymptote.

(g) [3 marks] Sketch the curve of $y = f(x)$.